On some interpolation problems

A. Gombani
LADSEB-CNR
Corso Stati Uniti 4
35127 Padova, Italy
gombani@ladseb.pd.cnr.it

Gy. Michaletzky
Eötvös Loránd University
H-1111 Pázmány Péter sétány 1/C,
Computer and Automation Institute of HAS
H-1111 Kende u. 13-17
Budapest, Hungary
michgy@ludens.elte.hu

Abstract

In this paper we investigate some aspect of the partial realization problem formulated for Schur-functions considered on the right half plane of $\mathbb{C}$. This analysis can be considered to be partially complementary to the results of A. Lindquist, C. Byrnes et al. on Carathéodory functions, [2], [4], [3].

1 Preliminaries and notation

Let F be a rational $p \times m$ matrix of McMillan degree N, whose entries lie in the Hardy space of the right half-plane. We shall denote by $\mathbb{C}^+$ the right half-plane, and by $\mathcal{H}^2_+$ the corresponding Hardy space of vector or matrix valued functions (the proper dimension will be understood from the context). The space $\mathcal{H}^2_+$ is naturally endowed with the scalar product,

$$< F, G >= \frac{1}{2\pi}\mathrm{Tr} \int_{-\infty}^{\infty} F(iy)G(iy)^*\, dy, \tag{1.1}$$

and we shall denote by $\| \ \|_2$ the associated norm. Note that if M is a complex matrix, Tr stands for its trace, M^T for its transpose and M^* for its transpose conjugate. Similarly, we define $\mathcal{H}^\infty_+$ to be Hardy space of essentially bounded functions analytic on the right half plane.

We assume that we are given a set of interpolation points $s_1, ..., s_n$ in the right half-plane $\mathbb{C}^+$ and interpolating conditions

$$U = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \qquad V = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \tag{1.2}$$

with u_i, v_i row vectors in $\mathbb{C}^p$, $\|u_i\| = 1$ and $\|v_i\| < 1$ for $i = 1, ..., n$ and we want to find the solutions Q to the problem

$$u_i Q(s_i)^* = v_i \qquad i = 1, ..., n \tag{1.3}$$

which are Schur-functions.

It is well-known (see eg. [1]) that all solutions of this problem can be given using a rational fractional representation defined by a J-inner function. This representation is even valid for a more general form of the interpolation problem (1.3) which allows for multiplicities of the interpolation nodes. This can be defined in the following way.

Problem 1.1. *Given the matrix $\mathcal{A}$ of size $n \times n$ with eigenvalues in the left half plane $\mathbb{C}^-$, and matrices U, V of size $n \times p$ and a constant matrix D of size $p \times p$ parameterize all Schur-functions Q of McMillan-degree at most n satisfying*

$$\begin{cases} (Q(s)U^* - V^*)\,(sI + \mathcal{A}^*)^{-1} & \text{is analytic on} \quad \mathbb{C}^+ \\ Q(\infty) = D\ . \end{cases} \tag{1.4}$$

In this case the eigenvalues of $-\mathcal{A}^*$ determine the interpolation nodes.

To avoid pathological cases, we assume that the functions $Q(s)u_i^*$ are non constant. Notice that this assumption is always satisfied if for all $i = 1, ..., n$ we have $u_i D^* \neq v_i$.

Among the solutions of the Problem 1.1 we would like to consider solutions with "low complexity". In the scalar case – $p = 1$ – this can be formulated as solutions with McMillan degree no greater than n. In the multivariate case the condition should be formulated in terms of "global zero structure", as follows.

Problem 1.2. *Given the matrix $\mathcal{A}$ of size $n \times n$ with eigenvalues in the left half plane $\mathbb{C}^-$, and matrices U, V of size $n \times p$ and a constant contractive matrix D of size $p \times p$, such that $[D, I]\begin{bmatrix} U^* \\ -V^* \end{bmatrix}$ is of full row rank, parameterize all functions Q for which*

(i) Q is a Schur-function;

(ii)

$$\begin{cases} (Q(s)U^* - V^*)\,(sI + \mathcal{A}^*)^{-1} & \text{is analytic on} \quad \mathbb{C}^+ \\ Q(\infty) = D\ . \end{cases} \tag{1.5}$$

(iii)

$$\left(\begin{bmatrix} U^* \\ -V^* \end{bmatrix}, -\mathcal{A} \right) \tag{1.6}$$

determine a global right null pair of the function $[Q(z), I]$.

Together with the interpolation conditions formulated in Problem 1.2 we are going to analyze solutions satisfying the following zero interpolation conditions

$$[I - Q(t_i)Q^*(t_i)]z_i = 0, \qquad i = 1, ..., n\ , \tag{1.7}$$

where $t_1, \ldots, t_n \in \mathbb{C}^-$, $z_1, \ldots, z_n \in \mathbb{C}^p$, $\|z_i\| = 1$, $i = 1, \ldots, n$.

The general form of this zero condition is the following:

$$(I - QQ^*)\, Q_Z \quad \text{and} \quad Q_Z \quad \text{have no common poles,} \tag{1.8}$$

where Q_Z is an inner function. In this case the poles of Q_Z determine the nodes of the zero conditions.

The following lemma shows that without loss of generality we can assume that $D = 0$.

Lemma 1.1. *Let Q be a Schur function, $D = Q(\infty)$ assuming that $DD^* < I$. Set*

$$\begin{aligned} Q_D &= (I - DD^*)^{1/2}\,(I - QD^*)^{-1}\,(Q - D)\,(I - D^*D)^{-1/2} \\ U_D &= (U - VD)\,(I - D^*D)^{-1/2} \\ V_D &= (V - UD^*)\,(I - DD^*)^{-1/2} \end{aligned}$$

Then Q_D is an Schur-function and Q is a solution of the interpolation problem (1.4) defined by $\mathcal{A}$, U, V and D if and only if Q_D is a solution of (1.4) defined by $\mathcal{A}$, U_D, V_D and 0.

2 State-space realizations of the solutions

The $J-$inner function Θ generating all solutions of the interpolation Problem 1.1 without constraints on McMillan-degree has the realization

$$\Theta = \left(\begin{array}{c|cc} \mathcal{A} & U & V \\ \hline -\,U^*\mathcal{P}^{-1} & I & 0 \\ V^*\mathcal{P}^{-1} & 0 & I \end{array}\right) \tag{2.1}$$

where $\mathcal{P}$ satisfies

$$\mathcal{A}\mathcal{P} + \mathcal{P}\mathcal{A}^* + UU^* - VV^* = 0 \tag{2.2}$$

Then the Schur-function Q is a solution of the interpolation Problem 1.1 if and only if

$$Q := (S\Theta_{12} + \Theta_{22})^{-1}(S\Theta_{11} + \Theta_{21})$$

where

$$\Theta = \begin{pmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{pmatrix} \tag{2.3}$$

and S is a Schur-function with $S(\infty) = D$.

Especially, if u_i and v_i are the rows of U and V and

$$\mathcal{A} = diag\{-\overline{s}_1, ..., -\overline{s}_n\}\ ,$$

then the Schur function Q is a solution to then Nevanlinna-Pick problem

$$Q(s_i)u_i^* = v_i^*\ .$$

Similarly, if we take $\mathcal{A}$ lower triangular and $\mathcal{P}$ diagonal, we obtain a Potapov factorization of Θ and therefore a Schur problem.

Lemma 2.1. *Let Θ be a $J-$inner function as in (2.1), and let* $S = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$. *Then Q has realization:*

$$Q = \left(\begin{array}{cc|c} \mathcal{A} + V(DU^* - V^*)\mathcal{P}^{-1} & -VC & -U + VD \\ -BU^*\mathcal{P}^{-1} & A & -B \\ \hline (DU^* - V^*)\mathcal{P}^{-1} & -C & D \end{array}\right) \tag{2.4}$$

Proposition 2.1. *If the pair (A, B) is controllable then the realization of the function Q defined in (2.4) is controllable, as well.*

Proof. The identity

$$\mathcal{A} + V(DU^* - V^*)\mathcal{P}^{-1} = -\mathcal{P}\mathcal{A}^*\mathcal{P}^{-1} + (VD - U)\, U^*\mathcal{P}^{-1}$$

implies that the controllability subspace defined by the pair

$$\left(\begin{bmatrix} \mathcal{A} + V(DU^* - V^*)\mathcal{P}^{-1} & -VC \\ -BU^*\mathcal{P}^{-1} & A \end{bmatrix}, \begin{bmatrix} -U + VD \\ -B \end{bmatrix}\right)$$

coincide with that determined by

$$\left(\begin{bmatrix} -\mathcal{P}\mathcal{A}^*\mathcal{P}^{-1} & -VC \\ 0 & A \end{bmatrix}, \begin{bmatrix} -U + VD \\ -B \end{bmatrix}\right)$$

To prove controllability the celebrated P-B-H test will be applied. The orthogonal complement of the controllability subspace is invariant under the adjoint of the state-transition matrix. Thus we can consider an eigenvector belonging to that subspace.

$$[\alpha^*, \beta^*] \begin{bmatrix} U - VD \\ B \end{bmatrix} \;=\; 0 \tag{2.5}$$

$$[\alpha^*, \beta^*] \begin{bmatrix} -\mathcal{P}\mathcal{A}^*\mathcal{P}^{-1} & -VC \\ 0 & A \end{bmatrix} \;=\; \lambda\, [\alpha^*, \beta^*] \tag{2.6}$$

The equation (2.6) gives that

$$\lambda\alpha^* \;=\; -\alpha^*\mathcal{P}\mathcal{A}^*\mathcal{P}^{-1}\;, \tag{2.7}$$

$$\lambda\beta^* \;=\; -\alpha^*VC + \beta^*A\;. \tag{2.8}$$

If $\alpha \neq 0$ then λ should be an eigenvalues of $-\mathcal{A}^*$, i.e. λ coincides with one of the interpolation nodes. In particular, Re $\lambda > 0$. Consequently, λ is not an eigenvalue of A, thus the matrix $\lambda I - A$ is nonsingular implying that

$$\beta^* = -\alpha^*VC\,(\lambda I - A)^{-1}\;. \tag{2.9}$$

Substituting this back to the equation in (2.5)

$$\alpha^* \left[U - V \left(D + C(\lambda I - A)^{-1} B\right)\right] = 0 \ . \tag{2.10}$$

Shortly,

$$\alpha^* \left[U - VS(\lambda)\right] = 0 \tag{2.11}$$

Observe that since λ is in the right half plane the inequality

$$S(\lambda)S(\lambda)^* \leq I \tag{2.12}$$

holds true.

Multiplying the Lyaponuv-equation (2.2) by α^* and α from the left and right, respectively, we get that

$$-2\mathrm{Re}\ \lambda\alpha^*\mathcal{P}\alpha + \alpha^* V \left[S(\lambda)S(\lambda)^* - I\right] V^*\alpha = 0$$

If α is nonzero then the first term is strictly negative while the second term is nonpositive leading to a contradiction.

If $\alpha = 0$ then equations in (2.5) and (2.6) give that

$$\begin{aligned} \beta^* B &= 0 \\ \beta^* A &= \lambda\beta^* \ . \end{aligned}$$

Now the controllability of the pair A, B implies that $\beta = 0$, concluding our proof. ∎

Proposition 2.2. *If the realization of S is observable, then the unobservability subspace of the realization (2.4) is determined by the range of the matrix $\begin{bmatrix} \alpha \\ \beta \end{bmatrix}$, where α and β are solutions of the equation*

$$\mathcal{A}\alpha = \alpha\Gamma \tag{2.13}$$

$$\left(S(s)U^* - V^*\right)\mathcal{P}^{-1}\alpha \left(sI - \Gamma\right)^{-1} = C\left(sI - A\right)^{-1}\beta \ . \tag{2.14}$$

Especially, in this case the dimension of the unobservability subspace in the realization of Q given by (2.4) is at most n.

Proof. Obviously the unobservability subspace of the realization given in (2.4) is determined by the pair

$$\left(\left[\left(DU^* - V^*\right)\mathcal{P}^{-1}, -C\right], \begin{bmatrix} \mathcal{A} & 0 \\ -BU^*\mathcal{P}^{-1} & A \end{bmatrix} \right) \ .$$

Assume that the column vectors of a matrix $\begin{bmatrix} \alpha \\ \beta \end{bmatrix}$ determine a basis in the unobservability subspace. Then

$$\left(DU^* - V^*\right)\mathcal{P}^{-1}\alpha - C\beta \ = \ 0 \tag{2.15}$$

$$\begin{bmatrix} \mathcal{A} & 0 \\ -BU^*\mathcal{P}^{-1} & A \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \ = \ \begin{bmatrix} \alpha \\ \beta \end{bmatrix}\Gamma \tag{2.16}$$

for some matrix Γ.

If a vector γ is in the kernel space of α, i.e. $\alpha\gamma = 0$, then the equation in (2.16) implies that

$$\alpha\Gamma\gamma = \mathcal{A}\alpha\gamma = 0,$$

thus $\Gamma\gamma \in \mathrm{Ker}(\alpha)$, so the kernel subspace of α is Γ invariant. Especially, it contains an eigenvector of Γ. In other words there exists a vector γ, for which the identities

$$\begin{aligned} \alpha\gamma &= 0 \\ \Gamma\gamma &= \mu\gamma \end{aligned}$$

for some μ hold. Multiplying the equations (2.15) and (2.16) from the right by γ we obtain that

$$\begin{aligned} C\beta\gamma &= 0 \\ A\beta\gamma &= \mu\beta\gamma \ . \end{aligned}$$

Thus the vector $\beta\gamma$ is in the unobservability subspace determined by the pair (C, A). Using the assumed observability of the realization S we get that $\beta\gamma = 0$. Consequently

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix} \gamma = 0 \ .$$

But the column vectors of the matrix $\begin{bmatrix} \alpha \\ \beta \end{bmatrix}$ are linearly independent, thus $\gamma = 0$.

The equation (2.16) gives that

$$\mathcal{A}\alpha = \alpha\Gamma \ ,$$

$$-BU^*\mathcal{P}^{-1}\alpha + A\beta = \beta\Gamma \ ,$$

thus

$$(sI - A)^{-1} BU^*\mathcal{P}^{-1}\alpha + \beta = (sI - A)^{-1} \beta(sI - \Gamma) \ .$$

Multiplying from the left by C and using equation (2.15) we obtain that

$$\left(D + C\,(sI - A)^{-1} B\right) U^*\mathcal{P}^{-1}\alpha - V^*\mathcal{P}^{-1}\alpha = C\,(sI - A)^{-1} \beta(sI - \Gamma) \ .$$

In other words

$$(S(s)U^* - V^*)\,\mathcal{P}^{-1}\alpha\,(sI - \Gamma)^{-1} = C\,(sI - A)^{-1} \beta \ , \tag{2.17}$$

proving (2.13) and (2.14). At the same time (2.13) implies the last part of the proposition. ■

Let us introduce the notations:

$$\begin{aligned} n &= \text{McMillan degree of } \Theta \ , \\ n_S &= \text{McMillan degree of } S \ , \\ n_Q &= \text{McMillan degree of } Q \ , \\ d_{o,S} &= \text{dimension of the unobservability subspace of the realization (2.4)} \ . \end{aligned}$$

Corollary 2.1. *Assume that the realization of the function S is minimal. Then*

$$n_Q = n + n_S - d_{o,S} \tag{2.18}$$

Moreover

$$n_Q \geq n_S \ .$$

Observe that the poles of the function on the right hand side of (2.14) are among the eigenvalues of the matrix A, while the function standing on the left hand side has formally poles at the eigenvalues of A and Γ. Thus the eigenvalues of this latter one should be cancelled. This is again an interpolation type condition where the interpolation nodes are defined by the eigenvalues of Γ forming a subset of the eigenvalues of $\mathcal{A}$. These interpolation nodes are in the left half plane $\mathbb{C}^-$. Note that in the special case when

$$n_Q \leq n$$

the number of the interpolation constraints formulated above on the function S should be as large as its McMillan-degree n_S.

Especially, if $\mathcal{A} = \operatorname{diag}(-\bar{s}_1, \ldots, -\bar{s}_n)$ and $n_Q = n_S$ then these interpolation conditions can be expressed as

$$S(-\bar{s}_j)U^* \mathcal{P}^{-1} e_j = V^* \mathcal{P}^{-1} e_j \ , \qquad i = 1, \ldots, n \ , \tag{2.19}$$

where e_j denotes the j-th unit vector.

3 The inverse transformation

Suppose now we invert the relation $Q = T_\Theta(S)$ and want to determine the degree. We have the following:

Lemma 3.1. *Let Θ be a $J-$inner function as in (2.1) and let*

$$Q = \left(\begin{array}{c|c} A_Q & B_Q \\ \hline C_Q & D_Q \end{array} \right) . \tag{3.1}$$

Then $S = T_\Theta^{-1}(Q)$ has the following non-minimal realization:

$$S = \left(\begin{array}{cc|c} A_Q & -B_Q U^* \mathcal{P}^{-1} & B_Q \\ V C_Q & \mathcal{A} + (U - V D_Q) U^* \mathcal{P}^{-1} & V D_Q - U \\ \hline C_Q & (V^* - D_Q U^*) \mathcal{P}^{-1} & D_Q \end{array} \right) . \tag{3.2}$$

In the case when Q is a solution of the interpolation Problem (1.4) then the state space of the realization (3.2) can be further reduced.

Proposition 3.1. *Let Q be a solution of the interpolation Problem (1.4) with realization*

$$Q = \left(\begin{array}{c|c} A_Q & B_Q \\ \hline C_Q & D_Q \end{array} \right) .$$

Assume that (C_Q, A_Q) is an observable pair. Then $S = T_\Theta^{-1}(Q)$ has the observable realization

$$S = \left(\begin{array}{c|c} A_Q - Y \mathcal{P}^{-1} V C_Q & B_Q - Y \mathcal{P}^{-1} (V D_Q - U) \\ \hline C_Q & D_Q \end{array} \right) , \tag{3.3}$$

where Y is defined by the equation

$$(Q(s) U^* - V^*) (sI + \mathcal{A}^*)^{-1} = C_Q (sI - A_Q)^{-1} Y . \tag{3.4}$$

Proof. Since the function $(Q(s)U^* - V^*)(sI + \mathcal{A}^*)^{-1}$ is analytic on $\mathbb{C}^+$, it might have poles only at the eigenvalues of A_Q, and vanishes at ∞, the observability of (C_Q, A_Q) implies the existence of the matrix Y in (3.4).

Evaluating the identity

$$Q(s)U^* - V^* = DU^* - V^* + C_Q (sI - A_Q)^{-1} B_Q U^* = C_Q (sI - A_Q)^{-1} \beta (sI + \mathcal{A}^*)$$

at ∞ the equation

$$DU^* - V^* = C_Q Y \tag{3.5}$$

is obtained. Subtracting $C_Q Y$ from both sides we get that

$$C_Q (sI - A_Q)^{-1} B_Q U^* = C_Q (sI - A_q)^{-1} (Y \mathcal{A}^* + A_Q Y) .$$

Observability of (C_Q, A_Q) implies that

$$B_Q U^* = Y \mathcal{A}^* + A_Q Y . \tag{3.6}$$

Let us apply the state transformation defined by the matrix

$$T = \begin{bmatrix} I & Y \\ 0 & \mathcal{P} \end{bmatrix}$$

for the realization (3.2). Using that

$$\begin{aligned}
\left[C_Q, \left(V^* - D_Q U^*\right) \mathcal{P}^{-1}\right] \begin{bmatrix} I & Y \\ 0 & \mathcal{P} \end{bmatrix} &= \left[C_Q, 0\right] \\
\begin{bmatrix} I & Y \\ 0 & \mathcal{P} \end{bmatrix} \begin{bmatrix} B_Q \\ VD - U \end{bmatrix} &= \begin{bmatrix} B_Q - Y\mathcal{P}^{-1}\left(VD - U\right) \\ \mathcal{P}^{-1}\left(VD - U\right) \end{bmatrix} \\
\begin{bmatrix} A_Q & -B_Q U^* \mathcal{P}^{-1} \\ VC_Q & \mathcal{A} + \left(U - VD\right) U^* \mathcal{P}^{-1} \end{bmatrix} \begin{bmatrix} I & Y \\ 0 & \mathcal{P} \end{bmatrix} &= \begin{bmatrix} I & Y \\ 0 & \mathcal{P} \end{bmatrix} \begin{bmatrix} A_Q - Y\mathcal{P}^{-1} V C_Q & 0 \\ \mathcal{P}^{-1} V C_Q & -\mathcal{A}^* \end{bmatrix}
\end{aligned}$$

the following realization is obtained:

$$S = \left(\begin{array}{c|c} A_Q - Y\mathcal{P}^{-1} V C_Q & B_Q - Y\mathcal{P}^{-1}\left(VD_Q - U\right) \\ \hline C_Q & D_Q \end{array} \right) . \tag{3.7}$$

The realization (3.3) is obviously observable if (C_Q, A_Q) is an observable pair. This concludes the proof of the proposition. ■

Now let us analyze the controllability subspace of the realization defined in (3.3).

Proposition 3.2. *Assume that the realization of Q given in (3.1) is minimal. If the functions $VQ(s) - U$ and $sI - \mathcal{A}$ have no common left zero-functions then the realization of S in (3.3) is minimal.*

Proof. Let us apply the P-H-B test. Assume that there exist a nonzero column vector α_Q and $\mu \in \mathbb{C}$ such that

$$\alpha_Q^* \left(B_Q - Y\mathcal{P}^{-1}\left(VD - U\right)\right) = 0 \tag{3.8}$$

$$\alpha_Q^* \left(A_Q - Y\mathcal{P}^{-1} V C_Q\right) = \mu \alpha_Q^* . \tag{3.9}$$

It is immediately obtained that $\alpha_Q^* Y \neq 0$. Otherwise the equations $\alpha_Q^* B_Q = 0$, $\alpha_Q^* A_Q = \mu\alpha^*$ would hold, contradicting to the controllability of (A_Q, B_Q).

Computing (3.8) U^*–(3.9)Y we get that

$$\alpha_Q^* \left(B_Q U^* - Y\mathcal{P}^{-1} V D U^* + Y\mathcal{P}^{-1} U U^* - A_Q Y + Y\mathcal{P}^{-1} V C_Q Y\right) = -\mu \alpha_Q^* Y .$$

Using (3.5) and (3.6) we get that

$$\alpha_Q^* \left(Y\mathcal{A} - Y\mathcal{P}^{-1} V V^* + Y\mathcal{P}^{-1} U U^*\right) = -\mu \alpha_Q^* Y .$$

I.e.

$$\left(\alpha_Q^* Y \mathcal{P}^{-1}\right)\left(\mu I - \mathcal{A}\right) = 0 \tag{3.10}$$

On the other hand equations in (3.8) and (3.9) can be arranged into the following form

$$[\alpha_Q^*, -\alpha_Q^* Y \mathcal{P}^{-1}] \begin{bmatrix} A_Q - \mu I & B_Q \\ VC_Q & VD_Q - U \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} . \tag{3.11}$$

Equations (3.10) and (3.11) indicate that the functions $sI - \mathcal{A}$ and $VQ(s) - U$ have a common zero direction at $s = \mu$, contradicting to our assumption. Thus the realization of S given in (3.3) is controllable, concluding the proof of the proposition. ■

Introduce the notation

$$d_{c,Q} = \text{dimension of the controllability subspace of the realization (3.3).}$$

According to Proposition 3.1 the realization (3.3) is observable. Consequently

$$n_S = d_{c,Q} \ .$$

Corollary 3.1. *Assume that the realization (3.1) of the function Q is minimal, moreover the matrices $\mathcal{A}$ and A_Q have no common eigenvalues and*

$$Ker \left(\begin{bmatrix} VQ(s) - U \\ sI - \mathcal{A} \end{bmatrix} \right) = \{0\}$$

for all $s \in \mathbb{C}$. Then

$$n_S = n_Q \ .$$

4 Solutions of the interpolation problem with "low complexity"

Consider now the interpolation problem formulated in Problem 1.2.

Theorem 4.1. *The Schur-function Q is a solution of the interpolation Problem 1.2 if and only if it has the realization*

$$Q = \left(\begin{array}{c|c} -\mathcal{A}^* + BU^* & B \\ \hline DU^* - V^* & D \end{array} \right) . \tag{4.1}$$

Proof. Assume that $Q = \left(\begin{array}{c|c} A_Q & B_Q \\ \hline C_Q & D \end{array} \right)$. According to the assumption the matrix $[D, I] \begin{bmatrix} U^* \\ -V^* \end{bmatrix}$ is of full row rank. Consequently, $[D, I]$ can be extended into a nonsingular matrix

$$\begin{bmatrix} D & I \\ D_1 & I \end{bmatrix} .$$

Set

$$Q_e(z) = \begin{bmatrix} Q(z) & I \\ D_1 & I \end{bmatrix} .$$

Then using assumption (iii) in Problem 1.2 the pair $\left(\begin{bmatrix} U^* \\ -V^* \end{bmatrix}, -\mathcal{A}^* \right)$ determines a global right-null pair of the function Q_e of size $2p \times 2p$. On the other hand the global right-null pair of Q_e is determined by

$$\begin{bmatrix} D & I \\ D_1 & I \end{bmatrix}^{-1} \begin{bmatrix} C_Q \\ 0 \end{bmatrix}, A_Q - [B_Q, 0] \begin{bmatrix} D & I \\ D_1 & I \end{bmatrix}^{-1} \begin{bmatrix} C_Q \\ 0 \end{bmatrix} .$$

Thus there exists an invertible transformation T such that

$$\begin{bmatrix} D & I \\ D_1 & I \end{bmatrix}^{-1} \begin{bmatrix} C_Q \\ 0 \end{bmatrix} T = \begin{bmatrix} U^* \\ -V^* \end{bmatrix}, \tag{4.2}$$

$$\left(A_Q - [B_Q, 0] \begin{bmatrix} D & I \\ D_1 & I \end{bmatrix}^{-1} \begin{bmatrix} C_Q \\ 0 \end{bmatrix} \right) T = T\left(-\mathcal{A}^*\right) , \tag{4.3}$$

giving that

$$\begin{aligned} A_Q T - B_Q U^* &= -T\mathcal{A}^* , \\ DU^* - V^* &= C_Q T . \end{aligned}$$

Thus

$$Q = \left(\begin{array}{c|c} T\left(-\mathcal{A}^* + T^{-1} B_Q U^*\right) T^{-1} & B_Q \\ \hline \left(DU^* - V^*\right) T^{-1} & D \end{array} \right) = \left(\begin{array}{c|c} -\mathcal{A}^* + T^{-1} B_Q U^* & T^{-1} B_Q \\ \hline DU^* - V^* & D \end{array} \right) , \tag{4.4}$$

which introducing the notation $B = T^{-1} B_Q$ gives eqution (4.1). Straightforward calcuation gives the converse statement. ∎

In this case the corresponding Schur-function S has the realization

$$S = \left(\begin{array}{c|c} \mathcal{A} + B_S U^* \mathcal{P}^{-1} & B_S \\ \hline \left(DU^* - V^*\right) \mathcal{P}^{-1} & D \end{array} \right) , \tag{4.5}$$

where

$$B_S = \mathcal{P} B + U - V D .$$

Note that any function with realization defined in (4.1) provides a solution of the interpolation problem (1.4) but the required stability of Q does not necessarily hold.

The following theorem shows that Theorem 4.1 can be formulated in a way which is similar to the form given in the so-called Kimura-Georgiou parameterization of all solutions of the Carathéodory interpolation problem with McMillan-degree no greater than the number of the interpolation conditions. [8], [6].

Theorem 4.2. *Consider the interpolation Problem 1.1 and denote by Ξ the inner function determined by the interpolation nodes and directions, i.e.*

$$\Xi(s) = I - U^* \mathcal{R}^{-1} (sI - \mathcal{A})^{-1} U \ ,$$

where $\mathcal{R}$ is the solution of the Ljapunov-equation

$$\mathcal{A}\mathcal{R} + \mathcal{R}\mathcal{A}^* + UU^* = 0 \ .$$

Then the Schur-function Q is a solution of the interpolation Problem 1.2 if and only if it can be written in the form

$$Q = G^* (F^*)^{-1} \ ,$$

where G, F are stable rational functions and ΞF^* is unstable, $F(\infty) = I$.

Proof. Assume that Q has the realization given in 4.1. Set

$$F^* = I - U^* (sI + \mathcal{A}^*)^{-1} B \ .$$

Then ΞF^* is a stable function and

$$QF^* = D - V^* (sI + \mathcal{A}^*)^{-1} B \ ,$$

which is an unstable function.

Conversely, if ΞF^* is a stable function, for which $F(\infty) = I$ then F^* has the form

$$F^* = I - U^* (sI + \mathcal{A}^*)^{-1} B$$

for some matrix B. Now if $Q = G^* (F^*)^{-1}$ is a solution then

$$\begin{aligned} G^* - \left(D - V^* (sI + \mathcal{A}^*)^{-1} B\right) &= QF^* - \left(D - V^* (sI + \mathcal{A}^*)^{-1} B\right) \\ &= Q - (QU^* - V^*) (sI + \mathcal{A}^*)^{-1} B - D \ . \end{aligned}$$

The first term should be an unstable function while the last term – using that Q is a solution of the interpolation Problem 1.1 – is a stable, strictly proper function. Thus it must be identically zero, proving that

$$G = D - V^* (sI + \mathcal{A}^*)^{-1} B \ .$$

Consequently the function Q has a realization given in (4.1). ■

Proposition 4.1. *Let Q be as in (4.1), and suppose it has McMillan-degree n. Then Q is stable if and only if*

$$B = \mathcal{P}_W^{-1}(U - W)$$

where $W \in \mathbb{C}^{n \times p}$ satisfies the generalized Pick condition, i.e. the Lyapunov equation

$$\mathcal{A}\mathcal{P}_W + \mathcal{P}_W\mathcal{A}^* + UU^* - WW^* = 0 \tag{4.6}$$

has a unique positive definite solution.

Proof. Assume that Q with the realization given in (4.1) is a stable function. The assumption concerning its McMillan-degree implies that this realization is minimal, i.e. there exists a matrix $\mathcal{P}_Q > 0$ for which the equation

$$(-\mathcal{A}^* + BU^*)\,\mathcal{P}_Q + \mathcal{P}_Q\,(-\mathcal{A} + UB^*) + BB^* = 0$$

holds. This can be arranged to

$$\mathcal{A}\mathcal{P}_Q^{-1} + \mathcal{P}_Q^{-1}\mathcal{A}^* + UU^* - \left(U + \mathcal{P}_Q^{-1}B\right)\left(U + \mathcal{P}_Q^{-1}B\right)^* = 0\,.$$

Introducing the notations

$$\begin{aligned} \mathcal{P}_W &= \mathcal{P}_Q^{-1} \\ W &= U + \mathcal{P}_Q^{-1}B \end{aligned}$$

the equation (4.6) is obtained.

Conversely assume that (4.6) holds. Straightforward calculation gives that

$$\begin{aligned} \mathcal{P}_W\left(-\mathcal{A}^* - \mathcal{P}_W^{-1}(U-W)U^*\right)\mathcal{P}_W^{-1} = \mathcal{A} + W\,(U^* - W^*)\,\mathcal{P}_W^{-1} \\ \left(\mathcal{A} + W\,(U^* - W^*)\,\mathcal{P}_W^{-1}\right)\mathcal{P}_W + \mathcal{P}_W\left(\mathcal{A} + W\,(U^* - W^*)\,\mathcal{P}_W^{-1}\right)^* + (U - W)\,(U^* - W^*) &= 0\,. \end{aligned}$$

Now the controllability of the pair

$$\left(\mathcal{A}^* + \mathcal{P}_W^{-1}\,(U - W)\,W^*, \mathcal{P}_W^{-1}(U - W)\right)$$

implies the stability of $\left(\mathcal{A} + W\,(U^* - W^*)\,\mathcal{P}_W^{-1}\right)$. ■

Theorem 4.3. *Let Q be as in (4.1), and suppose it has McMillan-degree n. Then Q is a Schur-function if and only if*

$$B = -R^{-1}\left(U - VD - \tilde{V}\,(I - D^*D)^{\frac{1}{2}}\right)$$

where $\tilde{V} \in \mathbb{C}^{n\times p}$ and $R \in \mathbb{C}^{n\times n}$, R is positive definite and the equation

$$\mathcal{A}R + R\mathcal{A}^* + UU^* - VV^* - \tilde{V}\tilde{V}^* = 0 \tag{4.7}$$

holds.

Proof. According to the positive real lemma the function Q is a Schur-function if and only if the equation

$$\begin{aligned} R\left(A_Q + B_Q D^*\,(I - DD^*)^{-1}\,C_Q\right) + \left(A_Q + B_Q D^*\,(I - DD^*)^{-1}\,C_Q\right)^* R \\ + C_Q^*\,(I - DD^*)^{-1}\,C_Q + RB_Q\,(I - D^*D)^{-1}\,B_Q^*R = 0 \end{aligned} \tag{4.8}$$

has a positive definite solution R. Expressing the matrices A_Q and C_Q using (4.1) straightforward calculation gives that

$$R\mathcal{A}^* + \mathcal{A}R + (UU^* - VV^*) - (RB + U - VD)(I - D^*D)^{-1}(B^*R + U^* - D^*V^*) = 0$$

holds. Introducing the notation

$$\tilde{V} = (RB + U - VD)(I - D^*D)^{-\frac{1}{2}}$$

equation (4.7) is obtained. ■

Based on this a parameterization of stable solutions to the interpolation problem can be obtained using the results in [5], [7].

Finally note that the right null-pair corresponding to the zeros of the function $I - QQ^*$ inside $\mathbb{C}^-$ is given by the matrices

$$\left(-\mathcal{A}^* + B(I - D^*D)^{-\frac{1}{2}}\tilde{V}^*, \quad V^* - (I - DD^*)^{-1}D(I - D^*D)^{\frac{1}{2}}\tilde{V}^*\right) .$$

Using the transformation allowed by Lemma 1.1 this has a particularly simple form:

$$\left(-\mathcal{A}^* + B\tilde{V}^*, \quad V^*\right) .$$

References

[1] J. A. Ball, I. Gohberg, and L. Rodman. Realization and interpolation of rational matrix functions. *Operator Theory, Advances and Applications*, 33:1–72, 1988.

[2] C. Byrnes and A. Lindquist. On the partial stochastic realization problem. *IEEE Trans. Automatic Control*, 42:1049–1070, 1997.

[3] C. Byrnes, A. Lindquist, and S. V. Gusev. A convex optimization approach to the rational covariance extension problem. *TRIA/MAT*, 1997.

[4] C. Byrnes, A. Lindquist, S. V. Gusev, and A. S. Mateev. A complete parametrization of all positive rational extension of a covariance sequence. *IEEE Trans. Automatic Control*, 40:1841–1857, 1995.

[5] L.Baratchart D. Alpay and A. Gombani. On the differential structure of matrix–valued rational inner functions. *Operator Theory: Advances and Applications*, 73:30–66, 1994.

[6] T. Georgiou. Partial realization of covariance sequences, 1983. Phd. Thesis.

[7] A. Gombani and M.Olivi. A new parametrization of rational inner functions of fixed degree: Schur parameters and realizations. *Mathematics of Control, Signals and Systems*, 13:156–177, 2000.

[8] H. Kimura. Positive partial realization of covariance sequences. In C. I. Byrnes and A. Lindquist, editors, *Modelling, Identification and Robust Control*, pages 499–513. Elsevier Science, 1986.