Non-regular processes and singular Kalman filtering

Augusto Ferrante, Giorgio Picci, Stefano Pinzoni
Dipartimento di Elettronica e Informatica
Università di Padova
via Gradenigo 6/A
35131 Padova, Italy
E-mail: augusto@dei.unipd.it, picci@dei.unipd.it, pinzoni@dei.unipd.it

Abstract

Contrary to the continuous-time case, a discrete-time process y can be represented by minimal linear models (see (1.1) below), which may either have a non-singular or a singular D matrix. In fact, models with $D = 0$ have been commonly used in the statistical literature. On the other hand, for models with a singular D matrix the Riccati difference equation of Kalman filtering involves in general the pseudo-inversion of a singular matrix. This "cheap filtering" problem has been studied for several decades in connection with the so-called "invariant directions" of the Riccati equation. For a singular D, a reduction in the order of the Riccati equation is in general possible. In this paper, we provide an explanation of this phenomenon from the classical point of view of "zero flippin" among minimal spectral factors. Changing D's occurs whenever zeros are "flipped" from $z = \infty$ to their reciprocals at $z = 0$. It is well known that for finite zeros the zero-flipping process takes place by multiplication of the underlying spectral factor by a suitable rational all-pass matrix function. For infinite zeros, zero-flipping is implemented by a dual version of the Silverman structure algorithm. Using this interpretation, we derive a new algorithm for filtering of non-regular processes, based on a reduced-order Riccati equation.

Keywords: Discrete-time processes, Stochastic realization, Riccati equation, Singular filtering, Silverman structure algorithm

1 Introduction

Consider a linear discrete-time stochastic model

$$
\begin{aligned}
x(t+1) &= Ax(t) + Bw(t) && \text{(1.1a)}\\
y(t) &= Cx(t) + Dw(t), && \text{(1.1b)}
\end{aligned}
$$

driven by a normalized p-dimensional white-noise process w. Any m-dimensional stationary process y admitting a representation of the form (1.1) has a spectral density matrix $\Phi(z)$, which is an $m \times m$ rational function of z. Representations of the type (1.1) are called *stochastic realizations* of the process y. *Minimal* stochastic realizations are such that the dimension of the state vector x is as small as possible: in this paper, we shall only deal with minimal stochastic realizations. Even assuming minimality, the representations (1.1) are highly non-unique. In fact, a fundamental result of stochastic system theory [2, 6] parametrizes the family of minimal stochastic realizations of a process with a given rational spectrum by the solutions of a certain linear matrix inequality, whose coefficients can be read off from a state-space realization of $\Phi(z)$. This matrix inequality reduces, in certain special instances, to an algebraic Riccati equation.

A general assumption, which we shall keep all through this paper, is that y is a *full-rank* process. This is equivalent to the spectral density matrix $\Phi(z)$ being of full rank, i.e., an invertible matrix, almost everywhere in z. As a consequence, in the model (1.1) the dimension of the process w is always greater than or equal to that of y, i.e., $p \geq m$. Now, it is well known that the same discrete-time process y can be represented by minimal realizations of the type (1.1), which may either have a non-singular or a singular D matrix. In fact, there may be realizations, like those used by Akaike in [1] and quite commonly encountered in the statistical literature, where $D = 0$.

When the matrix D in the representation (1.1) is singular, the problem of estimating the state x based on the (past) observations of y is known as "cheap (or singular) filtering". This problem is dual to the better known "cheap control" problem, and has been discussed in the literature for several decades, see [4, 13, 11], and references therein. It has been observed that, related to the singularity of D, there is a possible reduction in the order of the Riccati equation. This reduction has been investigated in a series of now classical papers by L. Silverman and co-authors (compare [22], and references therein), mostly in an optimal control context.

This paper is motivated by the observation that a straightforward dualization of the analysis of [22] does not apply naturally to the stochastic setting and in particular to the Riccati equation of the stochastic realization problem[1]. To the best of our knowledge, the question regarding the singularity of the D matrix in some (but in general not all) minimal realizations of a discrete-time process has been around for several decades in the literature, but has never been explained completely. In the recent paper [10], we provide an explanation of this phenomenon in the key of "zero flipping" among minimal spectral factors. The zeros which are "flipped" are zeros at $z = \infty$ being sent to their reciprocals at $z = 0$. By this procedure, one transforms spectral factors with a singular D matrix into other spectral factors with a "less singular" (and eventually non-singular) D. This process is implemented by a dual version of the Silverman structure algorithm which is described below. Using this interpretation, we point out that the reduction in the order of the ARE is related to a property of the process y (called *regularity*[2], see Definition 3.1 below), rather than to the singularity of the D matrix in a specific model. Moreover, we get a precise characterization of the amount of order reduction of the Riccati equation, which is afforded by zeros either at $z = \infty$ or at the origin, something which has traditionally been looked upon by studying "invariant

[1] In fact, past attempts in this direction [12] have been less than convincing.

[2] Not to be confused with the notion of *linear regularity* in [20].

directions".

The main contribution of the present paper is in Section 4, where we propose a reduced-order algorithm for filtering of non-regular processes. We show that if y is a non-regular process, the optimal filter may be derived by solving a reduced-order Riccati equation. The structure of this equation and the order reduction amount are analyzed and related to the parameters A, B, C, D of the given model and to the spectral density of the process y.

2 Background on stochastic realization

The material in this section is standard and can be found in various places in the literature [3, 7, 8]. We shall just recall the basic facts in order to set notations.

The transfer function $W(z) = C(zI - A)^{-1}B + D$ of any state-space representation of the process y of the type (1.1) is a *spectral factor* of $\Phi(z)$, i.e.,

$$\Phi(z) = W(z)W(z^{-1})^\top. \tag{2.1}$$

Note that two p-dimensional normalized white-noise processes w_1, w_2 differing by multiplication by a constant $p \times p$ orthogonal matrix are indistinguishable (as second order processes). Hence, it is natural to consider two realizations (1.1) with input noises differing by a constant orthogonal transformation as the same object. For this reason, we will not distinguish among spectral factors differing by right-multiplication by a constant $p \times p$ orthogonal matrix.

From now on, we shall fix our attention to *causal* realizations, where the matrix A has all eigenvalues strictly inside of the unit circle of the complex plane. The transfer function of each model (1.1) is then an *analytic* spectral factor of $\Phi(z)$, since it has no poles outside of the open unit disk, including the point $z = \infty$. To each such spectral factor of *minimal degree* (called a *minimal spectral factor*), we let correspond an equivalence class of minimal realizations (1.1), defined modulo a change of basis in the state space, an arbitrary $p \times p$ constant orthogonal transformation of the white-noise process w and, in the non-square case, the choice of some components of the noise process; see [16, 17, 18] for details. In this sense, the minimal causal realizations of y are essentially in a one-to-one correspondence with the (equivalence classes of) minimal analytic spectral factors $W(z)$.

If we decompose $\Phi(z) = \Phi(z^{-1})^\top$ into the analytic and co-analytic (with respect to the unit circle) components

$$\Phi(z) = \Phi_+(z) + \Phi_+(z^{-1})^\top, \tag{2.2}$$

then $\Phi_+(z)$ has a minimal realization whose parameters can be formally expressed as a function of the parameters of the model (1.1). In fact,

$$\Phi_+(z) = C(zI - A)^{-1}\bar{C}^\top + \tfrac{1}{2}\Lambda_0, \tag{2.3}$$

where $\bar{C}^\top = APC^\top + BD^\top$, P being the steady-state covariance $P := E\{x(t)x(t)^\top\}$, unique solution to the Lyapunov equation $P = APA^\top + BB^\top$. The matrix $\bar{C}$ must obviously be an invariant of the process (in the given basis), while Λ_0 in (2.3) is just the output covariance at lag zero, i.e.,

$$\Lambda_0 := E\{y(t)y(t)^\top\} = CPC^\top + DD^\top. \tag{2.4}$$

Well-known examples of minimal analytic spectral factors are the *outer*, also called *minimum-phase*, and the *maximum-phase* spectral factors, denoted $W_-(z)$ and $W_+(z)$, respectively. Both $W_-(z)$ and $W_+(z)$ are analytic in $\{z : |z| \geq 1\}$ including infinity, but while the outer factor has all zeros inside of the closed unit disk, $W_+(z)$ has all zeros outside of the open unit disk. The following result is standard [23, 16].

Theorem 2.1 *All minimal analytic rational spectral factors can be obtained by post-multiplying the minimum-phase factor $W_-(z)$ by a rational inner matrix function $Q(z)$, or by post-multiplying the maximum-phase factor $W_+(z)$ by a* co-analytic *rational inner matrix function $\bar{Q}(z)$, i.e., a rational matrix function analytic in $\{z : |z| < 1\}$, such that*

$$\bar{Q}(z)\bar{Q}(z^{-1})^\top = I. \tag{2.5}$$

Since the McMillan degree of minimal spectral factors has to be kept constant in the multiplication by the inner function, cancellation of zeros of $W_-(z)$ with poles of $Q(z)$, or cancellation of zeros of $W_+(z)$ with poles of $\bar{Q}(z)$ has to take place. Hence, some zeros are replaced by their reciprocal image with respect to the unit circle. This phenomenon is called "zero flipping" in the engineering literature. Zero flipping is closely related to solving a linear matrix inequality, as summarized in the following theorem, for the proof of which we refer, e.g., to [3, 8].

Theorem 2.2 *Let $(A, \bar{C}^\top, C, \frac{1}{2}\Lambda_0)$ be a minimal realization of the analytic component $\Phi_+(z)$ of the spectral density matrix $\Phi(z)$. Then, there is a one-to-one correspondence between minimal analytic spectral factors of $\Phi(z)$ and symmetric $n \times n$ matrices P solving the* Linear Matrix Inequality

$$M(P) := \begin{bmatrix} P - APA^\top & \bar{C}^\top - APC^\top \\ \bar{C} - CPA^\top & \Lambda_0 - CPC^\top \end{bmatrix} \geq 0. \tag{2.6}$$

In fact, corresponding to each solution $P = P^\top$ of (2.6), consider the unique (modulo orthogonal transformations) full column rank matrix factor $\begin{bmatrix} B \\ D \end{bmatrix}$ of $M(P)$,

$$M(P) = \begin{bmatrix} B \\ D \end{bmatrix} \begin{bmatrix} B^\top & D^\top \end{bmatrix}, \tag{2.7}$$

and define the rational matrix $W(z)$ parametrized in the form

$$W(z) = C(zI - A)^{-1}B + D. \tag{2.8}$$

Then, (2.8) is a minimal realization of a minimal analytic spectral factor of $\Phi(z)$.

Conversely, to each minimal analytic spectral factor $W(z)$ there corresponds, by suitably choosing a basis in the state space, a minimal realization of the form (2.8), for some B, D matrices. Then, the solution $P = P^\top$ of the Lyapunov equation $P - APA^\top = BB^\top$ satisfies the matrix equation (2.7) and hence the Linear Matrix Inequality (2.6).

Moreover, all symmetric solutions P of (2.6) are necessarily positive definite.

It can be shown [6, 8] that the set of solutions to the LMI (2.6)

$$\mathcal{P} := \{P \mid P = P^\top,\, M(P) \geq 0\} \tag{2.9}$$

is closed, bounded and convex. Moreover, there are two special elements P_-, $P_+ \in \mathcal{P}$, such that

$$P_- \leq P \leq P_+, \quad \text{for all } P \in \mathcal{P}, \tag{2.10}$$

where $P_1 \leq P_2$ means that $P_2 - P_1 \geq 0$, i.e., the difference $P_2 - P_1$ is a positive semidefinite matrix. To such minimal and maximal solutions of the LMI there correspond minimum-rank matrix factors $\begin{bmatrix} B_- \\ D_- \end{bmatrix}$ and $\begin{bmatrix} B_+ \\ D_+ \end{bmatrix}$ in the factorization (2.7), which yield the minimum- and maximum-phase spectral factors

$$W_-(z) = C(zI - A)^{-1}B_- + D_-, \qquad W_+(z) = C(zI - A)^{-1}B_+ + D_+, \tag{2.11}$$

respectively.

If $\Lambda_0 - CPC^\top > 0$, a simple calculation yields that $M(P) \geq 0$ if and only if P satisfies the algebraic Riccati inequality

$$P - APA^\top - (\bar{C}^\top - APC^\top)(\Lambda_0 - CPC^\top)^{-1}(\bar{C} - CPA^\top) \geq 0. \tag{2.12}$$

In particular, if P satisfies the algebraic Riccati equation

$$P = APA^\top + (\bar{C}^\top - APC^\top)(\Lambda_0 - CPC^\top)^{-1}(\bar{C} - CPA^\top), \tag{2.13}$$

the corresponding spectral factor $W(z)$ is *square* $m \times m$. The solutions $P = P^\top$ of (2.6) corresponding to square spectral factors form a subfamily $\mathcal{P}_0 \subset \mathcal{P}$. If $P \notin \mathcal{P}_0$, then $W(z)$ is rectangular $m \times p$, with $p > m$.

3 Regularity

One of the questions which naturally arise in the discrete-time context is for what kind of processes *all* minimal realizations have a *non-singular* D matrix. Here the word "non-singular" means that D is full row rank, i.e., D does possess a right-inverse. The following definition is from [6].

Definition 3.1 *The process y is* regular *if all its minimal realizations have a right-invertible D matrix.*

Right-invertibility of D is obviously equivalent to $R := DD^\top$ being non-singular and it can be seen that all system-theoretic properties of a discrete-time regular process are exactly the same as of a continuous-time process with a spectral density matrix strictly positive-definite at infinity. The following theorem collects some equivalent characterizations of the property of regularity.

Theorem 3.1 *Let y be a stationary process with a full-rank rational spectral density matrix $\Phi(z)$. Then, the following are equivalent.*

1. *The process y is regular.*
2. *For all solutions $P = P^\top$ of the LMI (2.6), $\Lambda_0 - CPC^\top > 0$.*
3. *$\Lambda_0 - CP_+C^\top > 0$, where $P_+ = P_+^\top$ is the maximal solution of the LMI (2.6) or, equivalently, $D_+ = W_+(\infty)$ is non-singular.*
4. *There exists a minimal spectral factor of $\Phi(z)$ having zeros neither at $z = 0$, nor at $z = \infty$.*
5. *All minimal spectral factors of $\Phi(z)h$ ave zeros neither at $z = 0$, nor at $z = \infty$.*
6. *The numerator matrix $\Gamma_- = A - B_- D_-^{-1} C$ of the minimum-phase spectral factor $W_-(z)$ is non-singular or, equivalently, $\lim_{z\to 0} W_-(z)^{-1}$ is finite.*
7. *$\Phi(z)$ has no zeros at infinity, nor at zero; more precisely, $\lim_{z\to\infty} \Phi(z)^{-1}$ is finite or, equivalently, $\lim_{z\to 0} \Phi(z)^{-1}$ is finite.*

While conditions 1 to 6 are more or less known, see [19, 14], condition 7 seems to be new. It states that the inverse $\Phi(z)^{-1}$ of the spectrum of a full-rank regular process is *proper*, i.e., it has no poles at $z = \infty$ (nor at $z = 0$). In [10] we provide a proof.

Regularity is quite restrictive. For instance, scalar processes admitting an AR representation

$$y(t) + \sum_{k=1}^{n} a_k y(t-k) = b_0 w(t), \tag{3.1}$$

with w normalized white noise and $a_n \neq 0$, cannot be regular if $n > 0$. Instead, MA processes described by models of the form

$$y(t) = \sum_{k=0}^{n} b_k w(t-k) \tag{3.2}$$

are regular. In fact, in the former case the spectral density function is

$$\Phi(z) = \frac{b_0^2}{(1+\sum_{k=1}^{n} a_k z^{-k})(1+\sum_{k=1}^{n} a_k z^{k})}, \tag{3.3}$$

with a zero at $z=\infty$ of multiplicity n, while in the second case we get

$$\Phi(z) = \left(\sum_{k=0}^{n} b_k z^{-k}\right)\left(\sum_{k=0}^{n} b_k z^{k}\right), \tag{3.4}$$

whose inverse is bounded as $z \to \infty$.

3.1 Zero flipping at infinity

We want now to analyze the minimal realizations (A, B, C, D) of a non-regular process y, with D a *singular*, i.e., not full row rank matrix. The following (quite obvious) lemma serves the purpose of linking singularity of D to the presence of zeros at infinity.

Lemma 3.1 *A proper rational matrix $W(z) = C(zI - A)^{-1}B + D$ has zeros at $z = \infty$ if and only if D is singular.*

Consider a minimal spectral factor $W(z) = C(zI - A)^{-1}B + D$ with a singular D. The "flipping" of zeros from $z=\infty$ to $z=0$ is accomplished by using a dual version of the well-known Silverman algorithm [**?**].

Assume the matrix D has p_0 linearly independent columns, with $0 \le p_0 \le m$. Let Q_0 be an orthogonal matrix such that $DQ_0 = \left[\begin{array}{c|c} D_{01} & 0 \end{array}\right]$, with $D_{01} \in \mathbb{R}^{m\times p_0}$ being full column rank. Let us partition $BQ_0 = \left[\begin{array}{c|c} B_{01} & B_{02} \end{array}\right]$ conformably, obtaining the following block structure,

$$W_0(z) := W(z)Q_0 = C(zI-A)^{-1}\left[\begin{array}{c|c} B_{01} & B_{02} \end{array}\right] + \left[\begin{array}{c|c} D_{01} & 0 \end{array}\right], \tag{3.5}$$

and let

$$\hat{W}_1(z) := W_0(z)\begin{bmatrix} I_{p_0} & 0 \\ 0 & zI_{p-p_0} \end{bmatrix}. \tag{3.6}$$

Clearly, $\hat{W}_1(z)$ is also a spectral factor of $\Phi(z)$. Since

$$\begin{aligned} \hat{W}_1(z) &= \left[\begin{array}{c|c} D_{01} + CB_{01}z^{-1} + CAB_{01}z^{-2} + \dots & CB_{02} + CAB_{02}z^{-1} + CA^2B_{02}z^{-2} + \dots \end{array}\right] \\ &= C(zI-A)^{-1}\left[\begin{array}{c|c} B_{01} & AB_{02} \end{array}\right] + \left[\begin{array}{c|c} D_{01} & CB_{02} \end{array}\right], \end{aligned}$$

this spectral factor has necessarily McMillan degree n and, hence, is also minimal. At this point, either $\left[\begin{array}{c|c} D_{01} & CB_{02} \end{array}\right]$ is right-invertible, or we may iterate the above procedure by introducing another orthogonal matrix Q_1, such that

$$\left[\begin{array}{c|c} D_{01} & CB_{02} \end{array}\right] Q_1 = \left[\begin{array}{c|c} D_{11} & 0 \end{array}\right],$$

with $D_{11} \in \mathbb{R}^{m \times p_1}$ of full column rank p_1, with $p_1 \geq p_0$, and define the minimal spectral factor

$$W_1(z) := \hat{W}_1(z)Q_1 = C(zI - A)^{-1} \left[\begin{array}{c|c} B_{11} & B_{12} \end{array} \right] + \left[\begin{array}{c|c} D_{11} & 0 \end{array} \right], \tag{3.7}$$

where $\left[\begin{array}{c|c} B_{11} & B_{12} \end{array} \right] = \left[\begin{array}{c|c} B_{01} & AB_{02} \end{array} \right] Q_1$.

Since y is a *full-rank* process, $W(z)$ as a rational function has full row rank m and hence, after a finite number of steps of the above procedure, we get a *minimal* spectral factor

$$W_l(z) := W(z)Q(z), \qquad Q(z) = Q_0 \prod_{i=0}^{l-1} \begin{bmatrix} I_{p_i} & 0 \\ 0 & zI_{p-p_i} \end{bmatrix} Q_{i+1}, \tag{3.8}$$

such that $W_l(\infty)$ is right-invertible, i.e., $W_l(z)$ has no zeros at infinity. Equivalently, $W_l(z)$ has a realization of the form

$$W_l(z) = C(zI - A)^{-1} \left[\begin{array}{c|c} B_{l1} & B_{l2} \end{array} \right] + \left[\begin{array}{c|c} D_{l1} & 0 \end{array} \right], \tag{3.9}$$

with D_{l1} square and invertible. In the following, we shall rename the transfer function $W_l(z)$ obtained at the last step of the Silverman algorithm as $W_S(z) = C(zI - A)^{-1} \left[\begin{array}{c|c} B_1 & B_2 \end{array} \right] + \left[\begin{array}{c|c} D_1 & 0 \end{array} \right]$.

So, the Silverman algorithm transforms a spectral factor $W(z)$ with a given zero structure at infinity to another, $W_S(z)$, which has no zeros at infinity. In fact, it can be shown that all zeros at infinity of $W(z)$ are replaced by corresponding zeros at $z = 0$ of $W_S(z)$, with the same multiplicity.

Finally, let

$$\Gamma := A - B_1 D_1^{-1} C \tag{3.10}$$

be the numerator matrix of $W_S(z)$ [15, 14] and consider the orthogonal complement of the column space of $\left[\begin{array}{c|c} \Gamma & B_2 \end{array} \right]$, which we may denote as Lker $\left[\begin{array}{c|c} \Gamma & B_2 \end{array} \right]$. In [10] we prove the following geometric characterization of regular processes.

Theorem 3.2 *The process y is regular if and only if*

$$\text{Lker} \left[\begin{array}{c|c} \Gamma & B_2 \end{array} \right] = \{0\}. \tag{3.11}$$

Clearly, this result is mostly useful when the D matrix in the given model is non-singular, i.e., $W_S(z) = W_0(z)$, and the process y might in principle be regular.

4 Steady-state filtering of non-regular processes: order-reduction of the ARE

In this section we shall consider steady-state estimation of the state x in a given model (1.1) for a non-regular observation process y. Such non-regular filtering problem encompasses (but is more general than) the *singular* filtering problem occurring when D is singular. Singular problems are usually addressed by writing the ARE with a Moore-Penrose pseudoinverse $\sharp$ in place of the usual inverse, i.e.,

$$X = AXA^\top - (AXC^\top + BD^\top)(CXC^\top + DD^\top)^\sharp(AXC^\top + BD^\top)^\top + BB^\top. \tag{4.1}$$

This formulation, however, hardly gives insight into the problem and may lead to substantially heavier computations than what is actually needed.

We shall show that the size of the ARE (4.1) associated with a non-regular observation process is always fictitiously large and that the problem complexity may be conveniently reduced even if D is non-singular. We would like to stress that the order reduction is a consequence of the non-regularity of the process y, rather than of the singularity of $DD^\top$. In fact, it does not depend on the particular realization (1.1) of y, but only on the process y itself. For this reason, the reduction procedure may by applied even in the standard (non-singular) filtering case.

It turns out that the reduction ν in the order of the ARE is an invariant of the process, equal to the sum of the multiplicities of the zeros of any model (1.1), located at $z = \infty$ or at $z = 0$. The order reduction ν can also be related to certain system theoretic properties of the matrices Γ and B_2 (cf. Theorem 3.2), that play a central rôle in stochastic realization theory [18] and smoothing estimation [9].

Let $W(z) = C(zI - A)^{-1}B + D$ be the transfer function of the given model (1.1). In general, D will not be right-invertible but, by using the dual Silverman algorithm of Section 3.1, we can always obtain an equivalent[3] model of y described by the transfer function

$$W_S(z) := W(z)Q(z) = C(zI - A)^{-1} \left[\begin{array}{c|c} B_1 & B_2 \end{array}\right] + \left[\begin{array}{c|c} D_1 & 0 \end{array}\right], \tag{4.2}$$

with D_1 non-singular. Here, the function $Q(z)$ is a polynomial conjugate-inner function given by the expression (3.8).

Let $P_0 = P_0^\top$ be the solution of the LMI (2.6) corresponding to the spectral factor $W_S(z)$, so that

$$P_0 - AP_0A^\top = B_1B_1^\top + B_2B_2^\top, \tag{4.3a}$$

$$\bar{C} = (AP_0C^\top + B_1D_1^\top)^\top, \qquad \Lambda_0 = D_1D_1^\top + CP_0C^\top, \tag{4.3b}$$

and let $P = P^\top$ be any solution of the LMI leading to a minimal *square* spectral factor with a *non-singular* D. One such solution of particular interest here is $P = P_-$, since the steady-state Kalman filter for the given model (1.1) is uniquely determined once P_- is known. In fact, the steady-state Kalman gain is given by

$$K = (\bar{C}^\top - AP_-C^\top)(\Lambda_0 - CP_-C^\top)^{-1}. \tag{4.4}$$

[3] "Equivalent" here means that $W_S(z)$ is also a minimal spectral factor of $\Phi(z)$. The state process of this model will in general be different from the original one.

However, the argument below will work for more general P's of the kind defined above. Any such P must satisfy

$$\Lambda_0 - CPC^\top > 0, \tag{4.5a}$$
$$P - APA^\top - (\bar{C}^\top - APC^\top)(\Lambda_0 - CPC^\top)^{-1}(\bar{C} - CPA^\top) = 0. \tag{4.5b}$$

Define $\Delta := P_0 - P$. By subtracting (4.5) from (4.3a), we get the algebraic Riccati equation

$$\Delta = A\Delta A^\top - (A\Delta C^\top + B_1 D_1^\top)(D_1 D_1^\top + C\Delta C^\top)^{-1}(A\Delta C^\top + B_1 D_1^\top)^\top + B_1 B_1^\top + B_2 B_2^\top, \tag{4.6}$$

which is the standard ARE satisfied by the steady-state error covariance matrix of the state estimate. Writing A as $\Gamma + B_1 D_1^{-1} C$, the ARE assumes the form

$$\Delta = \Gamma\Delta\Gamma^\top - \Gamma\Delta C^\top \left(D_1 D_1^\top + C\Delta C^\top\right)^{-1} C\Delta\Gamma^\top + B_2 B_2^\top. \tag{4.7}$$

The ARE (4.7) has a fictitiously large size. More precisely, in a suitable basis, any solution of (4.7) has the form

$$\Delta = \begin{bmatrix} \Delta_1 & 0 \\ 0 & 0 \end{bmatrix}, \tag{4.8}$$

where Δ_1 solves a reduced-order ARE (RARE). Theorem 4.1 below will describe the procedure to obtain such reduced-order ARE and also clarify what is the maximal amount of reduction that one can get.

As a preliminary step, select a square orthogonal matrix T, such that

$$\hat{\Gamma} := T\Gamma T^\top = \begin{bmatrix} \Gamma_R & \Gamma_{RI} & \Gamma_{RN} \\ 0 & \Gamma_I & \Gamma_{IN} \\ 0 & 0 & \Gamma_N \end{bmatrix}, \qquad \hat{B}_2 := TB_2 = \begin{bmatrix} B_{2R} \\ 0 \\ 0 \end{bmatrix}, \tag{4.9}$$

where the pair (Γ_R, B_{2R}) is reachable, Γ_I is invertible and $\Gamma_N \in \mathbb{R}^{\nu\times\nu}$ is nilpotent. Note that Γ_N is the nilpotent part of the map induced by Γ on the quotient space $\mathbb{R}^n/\langle\Gamma \mid B_2\rangle$. In other words, Γ_N describes the *invariant zero-dynamics at* $z = 0$ of $W_S(z)$.

Define

$$\Gamma_1 := \begin{bmatrix} \Gamma_R & \Gamma_{RI} \\ 0 & \Gamma_I \end{bmatrix} \in \mathbb{R}^{(n-\nu)\times(n-\nu)}, \qquad B_{21} := \begin{bmatrix} B_{2R} \\ 0 \end{bmatrix} \in \mathbb{R}^{(n-\nu)\times(p-m)}, \tag{4.10}$$

and partition $CT^\top$ as $CT^\top = \left[\begin{array}{c|c} C_1 & C_2 \end{array}\right]$, with C_1 of dimension $m \times (n-\nu)$. In [10] the following theorem is proved.

Theorem 4.1 *Let T be an $n \times n$ orthogonal matrix leading to the special form (4.9). Then, there is a bijective correspondence between the symmetric solutions of the ARE (4.7) and those of the RARE*

$$\Delta_1 = \Gamma_1\Delta_1\Gamma_1^\top - \Gamma_1\Delta_1 C_1^\top \left(D_1 D_1^\top + C_1\Delta_1 C_1^\top\right)^{-1} C_1\Delta_1\Gamma_1^\top + B_{21} B_{21}^\top, \tag{4.11}$$

given by

$$\Delta = T^\top \begin{bmatrix} \Delta_1 & 0 \\ 0 & 0 \end{bmatrix} T. \tag{4.12}$$

The RARE (4.11) has order $n - \nu$, with ν being the algebraic multiplicity of the invariant zero at $z = 0$ of $W_S(z)$. The ARE cannot be reduced further.

This theorem suggests the following procedure to reduce the order of the ARE and efficiently compute the steady-state Kalman filter gain for a non-regular process.

1. Apply the Silverman algorithm to $W(z)$ to get $W_S(z)$ with a non-singular D matrix.

2. Compute the state covariance matrix P_0 of the transformed model by solving the Lyapunov equation (4.3a).

3. Do an orthogonal change of basis on the realization of $W_S(z)$ (e.g., bring it to the real Schur form) to find T and Γ_1, C_1, B_{21}.

4. Find the maximal solution $\Delta_{1,\mathrm{MAX}}$ of the RARE (4.11).

5. Compute Δ_{MAX} using (4.12) and $P_- = P_0 - \Delta_{\mathrm{MAX}}$ to get the steady-state Kalman gain K from (4.4).

The reduction procedure is actually performed on the ARE relative to the model $W_S(z)$, obtained from the original model $W(z)$ by flipping the zeros at infinity into $z = 0$. In particular, if D is not right-invertible, it is guaranteed that the ARE can be somewhat reduced. However, we have seen that, as long as the process y is non-regular, such order reduction occurs even if the original D matrix is right-invertible.

Acknowledgment. This research was partially supported by the ministry of higher education of Italy (MIUR), under project *Identification and Adaptive Control of Industrial Systems.*

References

[1] H. Akaike. Markovian representation of stochastic processes by canonical variables. *SIAM Journal on Contr. Optim.*, 13:162–173, 1975.

[2] B. D. O. Anderson. A system theory criteria for positive real matrices. *SIAM Journal on Contr. Optim.*, 5:171–182, 1967.

[3] B. D. O. Anderson. The inverse problem of stationary covariance generation. *J. of Stat. Phys.*, 1:133–147, 1967.

[4] R. S. Bucy, D. Rappaport, and L. M. Silverman. Correlated noise filtering and invariant directions for the Riccati equation. *IEEE Trans. Automatic Control*, AC-15:535–540, 1970.

[5] C. Commault and J. M. Dion. Structure at infinity of linear multivariable systems: a geometric approach. *IEEE Trans. Automatic Control*, AC-27:693–696, 1982.

[6] P. Faurre. Realisations markoviennes de processus stationnaires. Technical Report 13, INRIA (LABORIA), Le Chesnay, France, March 1973.

[7] P. Faurre. Stochastic realization algorithms. In R. K. Mehra, editor, *System Identification: advances and case studies*, pp. 1–25. Academic Press, 1976.

[8] P. Faurre, M. Clerget, and F. Germain. *Opérateurs Rationnels Positifs*. Dunod, 1979.

[9] A. Ferrante and G. Picci. Minimal realization and dynamic properties of optimal smoothers. *IEEE Trans. Automat. Contr.*, AC-45:2028–2046, 2000.

[10] A. Ferrante, G. Picci, and S. Pinzoni. Silverman algorithm and the structure of discrete-time stochastic systems. *Linear Algebra and its Applications, special issue on System Theory*, 2002.

[11] T. Geerts. The Algebraic Riccati equation and singular optimal control: The discrete time case. In U. Helmke, R. Mennicken, and J. Saurer, editors, *Proc. MTNS 93.*, Systems and Networks: Mathematical Theory and Applications II, pp. 129–134. Akademie Verlag, 1994.

[12] A. Kitapci, E. Jonkheere and L. M. Silverman. Singular optimal filtering. In C. I. Byrnes and A. Lindquist, editors, *Frequency Domain and State Space Methods for Linear Systems*, Elsevier Science Publ. (North Holland), 1986.

[13] H. Kwakernaak and R. Sivan. *Linear Optimal Control Systems*. John Wiley & Sons, New York, 1972.

[14] A. Lindquist and Gy. Michaletzky. Output-induced subspaces, invariant directions and interpolation in linear discrete-time stochastic systems. *SIAM Journal on Contr. Optim.*, 35:810–859, 1997.

[15] A. Lindquist, Gy. Michaletzky, and G. Picci. Zeros of spectral factors, the geometry of splitting subspaces, and the algebraic Riccati inequality. *SIAM Journal on Contr. Optim.*, 33:365–401, 1995.

[16] A. Lindquist and G. Picci. On the stochastic realization problem. *SIAM Journal on Contr. Optim.*, 17:365–389, 1979.

[17] A. Lindquist and G. Picci. Realization theory for multivariate stationary Gaussian processes. *SIAM Journal on Contr. Optim.*, 23:809–857, 1985.

[18] A. Lindquist and G. Picci. A geometric approach to modelling and estimation of linear stochastic systems. *Journal of Mathematical Systems, Estimation, and Control*, 1:241–333, 1991.

[19] M. Pavon. Stochastic realization and invariant directions of the matrix Riccati equation. *SIAM Journal on Contr. Optim.*, 28:155–180, 1980.

[20] Y. A. Rozanov. *Stationary Random Processes*. Holden Day, S. Francisco, 1967.

[21] L. M. Silverman. Inversion of multivariable linear systems. *IEEE Trans. Automat. Contr.*, AC-14:270–276, 1969.

[22] L. M. Silverman. Discrete Riccati equation: alternative algorithms, asymptotic properties, and system theory interpretations. In C. T. Leondes, editor, *Control and Dynamic Systems*, pp. 313–386. Academic Press, New York, 1976.

[23] D. C. Youla. On the factorization of rational matrices. *IEEE Trans. P. I. T.*, 7:172–189, 1961.