Structural Properties of LTI Singular Systems under Output Feedback

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Abstract

Generic results regarding the structural properties of LTI singular system are presented in this paper. These include the structure of the system poles (both finite and infinite), assignment of the finite poles, elimination of the impulse mode, and controllability and observability of the closed-loop system. These properties are characterized by some new concepts defined in this paper.

1 Introduction

In the past two decades, a considerable amount of research concerning linear time-invariant (LTI) singular systems (or descriptor systems) has been reported because of their extensive applications. It is well known that a singular system has complicated structures and contains not only finite poles but also infinite pole which may generate undesired impulse behaviors [7]. Efforts have been devoted to investigating structural properties of this kind of singular systems. Results on controllability and observability, stability, pole assignment, and feedback regularization have been established by both algebraic and geometric approaches (see, for example, [2, 5] and the references therein). [1] shows that the finite poles can be freely assigned and the infinite pole can all be eliminated by state feedback if the system is strongly controllable (i.e., both R-controllable and impulse controllable). It is shown [15] under the assumption of impulse controllability and observability, the closed-loop system can be made impulse-free by almost any output feedback. [8] calculates the number of the infinite pole that are eliminable based-on the slow/fast decomposition of the system. [14] presents an equivalence of pole assignability between a strongly controllable and observable LTI singular system and a completely controllable and observable non-singular system.

This paper studies three basic problems regarding structural properties of singular systems. Namely, 1) the algebraic structures of both the finite poles and the infinite pole; 2) the assignability of the finite poles and the elimination of the infinite pole by output feedback; and 3) the controllability and observability of the system with minimal number of inputs and outputs. New generic solutions to these problems are presented in terms of the concepts defined in this paper.

2 Open-loop systems

Consider a linear time-invariant singular system

$$E\dot{x} = Ax + Bu, \ y = Cx \tag{1}$$

where $x \in \mathcal{R}^n$ is the state of the system, $u \in \mathcal{R}^r$ and $y \in \mathcal{R}^m$ are the input and output vectors of the system respectively, $E \in \mathcal{R}^{n\times n}$ is assumed to be singular with $0 <$rank$(E)< n$, and A, B, C are real constant matrices of appropriate sizes.

We assume in this section that the system (1) is regular, i.e., $\det(sE-A) \neq 0$. Let $\deg(\det(sE-A)) = n_1$, then the system (1) has n_1 finite poles (counted repeatedly for multiple poles), defined as the eigenvalues of the matrix pair (E, A). Let $\lambda \in \sigma(E, A) = \{s \mid \det(sE - A) = 0, \text{ finite } s \in \mathcal{C}\}$ be a finite pole of the system (1). Its geometric multiplicity (GM) is defined as $\mathrm{gm}(\lambda, E, A) = \dim[\mathrm{null}(\lambda E - A)]$.

Definition 1 *The finite cycle index (FCI) of the system (1) is defined as*

$$\mathrm{cyc}(E, A) = \max\{\dim[\mathrm{null}(sE - A)], \ \textit{finite } s \in \mathcal{C}\}$$

The system (1) has infinite pole (impulse mode) if $\deg(\det(sE - A)) < \mathrm{rank}(E)$. The algebraic multiplicity (number) of the impulse mode is defined as the degree deficiency, i.e., $\mathrm{alg}_\infty(E, A) = \mathrm{rank}(E) - \deg(\det(sE - A))$.

Let the Smith canonical form of E-λA be

$$E - \lambda A \approx \mathrm{diag}[\varphi_1(\lambda), \varphi_2(\lambda), ..., \varphi_\mu(\lambda), O]$$

where μ is the normal rank of $E - \lambda A$, and $\varphi_1(\lambda), ..., \varphi_\mu(\lambda)$ are polynomials of λ with leading coefficient one, satisfying

$$\varphi_1(\lambda)|\varphi_2(\lambda)| \cdots |\varphi_\mu(\lambda)$$

and O stands for the zero block matrix of appropriate size. The infinity rank of $sE - A$ is then defined in [13] as

$$\mathrm{rank}_\infty[sE - A] = \max\{k \mid \lim_{\lambda\to 0}\frac{\varphi_k(\lambda)}{\lambda} \neq 0, \text{ and } \lim_{\lambda\to 0}\frac{\varphi_{k+1}(\lambda)}{\lambda} = 0\}$$

Lemma 1 *[9]The infinity rank of matrix pencil $sE - A$ can be determined by*

$$\mathrm{rank}_\infty[sE - A] = \mathrm{rank}\begin{bmatrix} E & 0 \\ A & E \end{bmatrix} - \mathrm{rank}(E)$$

Definition 2 *The geometric multiplicity (GM) of the infinity pole of the system (1) is defined as*

$$\mathrm{gm}_\infty(E, A) = n - \mathrm{rank}_\infty[sE - A]$$

The GM of the infinite pole is also called impulse cycle index (ICI) of the system (1), and denoted by $\mathrm{cyc}_\infty(E, A)$*. The overall cycle index (OCI) of the system (1) is defined as*

$$\mathrm{Cyc}(E, A) = \max\{\mathrm{cyc}(E, A), \mathrm{cyc}_\infty(E, A)\}$$

Note that $0 \leq \mathrm{gm}_\infty(E, A) = \mathrm{cyc}_\infty(E, A) \leq \mathrm{alg}_\infty(E, A) \leq \mathrm{rank}(E)$.

Lemma 2 *The GM of the impulse mode (the ICI) of the system (1) is given by*

$$\mathrm{gm}_\infty(E, A) = \mathrm{cyc}_\infty(E, A) = n - \mathrm{rank}\begin{bmatrix} E & 0 \\ A & E \end{bmatrix} + \mathrm{rank}(E)$$

Proposition 1 *The following statements are true.*

(a) The system (1) has no impulse mode (impulse-free) if and only if $\mathrm{a\,lg}_{\infty}(E, A) = 0$ *(or* $\mathrm{gm}_{\infty}(E, A) = 0$*, or* $\mathrm{cyc}_{\infty}(E, A) = 0$*).*

(b) The system (1) has no finite pole if and only if $\mathrm{cyc}(E, A) = 0$*.*

3 Closed-loop systems

Apply static output feedback

$$u = Ky + v \tag{2}$$

to the system (1), we have the closed-loop system

$$E\dot{x} = (A + BKC)x + Bv,\ y = Cx \tag{3}$$

We now assume that the system (1) is regularizable by output feedback (2). That is, the system (3) is regular for some K.

The output fixed polynomial (FP) of the system (1) is defined as

$$p_f(s, E, A, B, C) = \gcd\left\{\det(sE - A - BKC), K \in \mathcal{R}^{r\times m}\right\}$$

where gcd stands for greatest common divisor. The roots of the FP is defined as the finite fixed modes (FFM) of the system (1), i.e.,

$$\Lambda(E, A, B, C) = \underset{K\in\mathcal{R}^{r\times m}}{\cap} \sigma(E, A + BKC)$$

If $\deg[\det(sE - A - BKC)] < \mathrm{rank}(E)$ for all $K \in \mathcal{R}^{r\times m}$, then the system (1) is said to have an impulse fixed mode (IFM).

The following definitions characterize the ability of the output feedback (2) to change the pole structures of the system (1).

Definition 3 *The output variable polynomial (VP) of the system (1) is defined as*

$$p_v(s, E, A, B, C, K) = \frac{\det(sE - A - BKC)}{p_f(s, E, A, B, C)}$$

Note that the zeros of the VP are the variable finite poles of the system (1).

Definition 4 *Let* $\lambda \in \Lambda(E, A, B, C)$*, its algebraic and geometric multiplicity are defined as, respectively,*

$$\mathrm{a\,lg}(\lambda, E, A, B, C) = \min\{\mathrm{a\,lg}(E, A + BKC), K \in \mathcal{R}^{r\times m}\}$$
$$\mathrm{gm}(\lambda, E, A, B, C) = \min\{\mathrm{gm}(E, A + BKC), K \in \mathcal{R}^{r\times m}\}$$

Definition 5 *The algebraic multiplicity (number) of the IFM is defined as*

$$\mathrm{a\,lg}_{\infty}(E, A, B, C) = \min\{\mathrm{a\,lg}(E, A + BKC), K \in \mathcal{R}^{r\times m}\}$$

And the GM of the IFM is defined as

$$\mathrm{gm}_{\infty}(E, A, B, C) = \min\left\{\mathrm{gm}_{\infty}(E, A + BKC), K \in \mathcal{R}^{r\times m}\right\}$$

Definition 6 *The finite output feedback cycle index (FOCI) of the system (1) is defined as*

$$\mathrm{cyc}(E,A,B,C) = \min\{\mathrm{cyc}(E,A+BKC),\ K \in \mathcal{R}^{r\times m})\}$$

Similarly, the impulse output feedback cycle index (IOCI) of the system (1) is defined as

$$\mathrm{cyc}_\infty(E,A,B,C) = \min\{\mathrm{cyc}_\infty(E,A+BKC), K \in \mathcal{R}^{r\times m}\}$$

And the overall output feedback cycle index (OOCI) of the system (1) is defined as

$$\mathrm{Cyc}(E,A,B,C) = \min\{\mathrm{Cyc}(E,A+BKC), K \in \mathcal{R}^{r\times m}\}$$

As immediate consequences of the above definitions, we have the following characterizations of the regularization and complete impulse mode elimination.

Proposition 2 *The following statements are true.*

(a) The impulse mode of the system (1) can all be eliminated by almost any output feedback (2) if and only if $\mathrm{a\,lg}_\infty(E,A,B,C) = 0$ *(or* $\mathrm{gm}_\infty(E,A,B,C) = 0$*, or* $\mathrm{cyc}_\infty(E,A,B,C) = 0$*).*

(b) The system (1) has no IFM if and only if $\mathrm{cyc}_\infty(E,A,B,C) = 0$.

It can be shown that it is generic that any output feedback matrix $K \in \mathcal{R}^{r\times m}$ will give the respective index. In other words, for almost any $K \in \mathcal{R}^{r\times m}$, we have, for example,

$$\begin{aligned} \mathrm{gm}(\lambda,E,A,B,C) &= \mathrm{gm}(\lambda,E,A+BKC) \\ \mathrm{a\,lg}_\infty(E,A,B,C) &= \mathrm{a\,lg}_\infty(E,A+BKC) \\ \mathrm{cyc}_\infty(E,A,B,C) &= \mathrm{cyc}_\infty(E,A+BKC) \end{aligned}$$

This means that these indices can be determined from the pair $(E, A+BKC)$ for almost any randomly selected matrix $K \in \mathcal{R}^{r\times m}$. Particularly, if two such selected matrices result in the same numbers, we can be practically sure that the numbers are the respective indices indeed.

It is easy to see that

$$\mathrm{Cyc}(E,A,B,C) = \max\{\mathrm{cyc}(E,A,B,C), \mathrm{cyc}_\infty(E,A,B,C)\}$$

In addition, the geometric multiplicity of the FFM and the IFM can be determined directly in terms of the system matrices as given below.

Theorem 1 *Let* $\lambda \in \Lambda(E,A,B,C)$*, then*

$$\mathrm{gm}(\lambda,E,A,B,C) = n - \min\{\mathrm{rank}\begin{bmatrix} \lambda E - A & B \end{bmatrix}, \mathrm{rank}\begin{bmatrix} \lambda E - A \\ C \end{bmatrix}\}$$

And

$$\mathrm{gm}_\infty(E,A,B,C) = \mathrm{cyc}_\infty(E,A,B,C) = n + \mathrm{rank}(E) - \min\{\mathrm{rank}\begin{bmatrix} E & 0 & 0 \\ A & B & E \end{bmatrix}, \mathrm{rank}\begin{bmatrix} E & 0 \\ A & E \\ C & 0 \end{bmatrix}\}$$

For the finite output feedback cycle index, we have that $\mathrm{cyc}(E, A, B, C) = 0$ when the closed-loop system (3) always has no finite pole.

Theorem 2 *Except the special case mentioned above, the FOCI of the system (1) can be determined as follows*

(a) If $\Lambda(E, A, B, C) \neq \Phi$*, then* $\mathrm{cyc}(E, A, B, C) = \max\{\mathrm{gm}(\lambda, E, A, B, C), \lambda \in \Lambda(E, A, B, C)\}$*;*

(b) Otherwise $\mathrm{cyc}(E, A, B, C) = 1$.

4 Assignment of the finite poles and elimination of impulse mode

Theorem 3 *The assignability of the variable finite poles of the system (1) is equivalent to that of a non-singular triple that is completely controllable and observable. Consequently, for almost any* $K \in \mathcal{R}^{r\times m}$*, the variable finite poles of the system (1) are distinct, and away from any given finite set in the complex plane.*

Theorem 4 *Consider the system (1). The number of impulse mode (counted repeatedly) that can be eliminated by output feedback (2) is given by*

$$a_e = \mathrm{a\,lg}_\infty(E, A) - \mathrm{a\,lg}_\infty(E, A, B, C)$$

The number of independent impulse mode that can be eliminated by output feedback (2) is given by

$$\begin{aligned} g_e &= \mathrm{gm}_\infty(E, A) - \mathrm{gm}_\infty(E, A, B, C) \\ &= \min\{\mathrm{rank}\begin{bmatrix} E & 0 & 0 \\ A & B & E \end{bmatrix}, \mathrm{rank}\begin{bmatrix} E & 0 \\ A & E \\ C & 0 \end{bmatrix}\} - \mathrm{rank}\begin{bmatrix} E & 0 \\ A & E \end{bmatrix} \end{aligned}$$

Moreover the elimination can be achieved by almost any $K \in \mathcal{R}^{r\times m}$.

The contribution of Theorem 4 is twofold. First, it gives solution to partial impulse mode elimination problem, including the impulse free and regularization problem as a special case. Secondly, it points out the generic nature of the solution.

5 Controllability and observability

The roles of the cycle indices in characterizing the controllability and observability of the singular system are given by the following results.

Proposition 3 *Assume that the system (1) is regular, then*

(a) it is R-controllable (R-observable) only if $\mathrm{col}(B) \geq k$ $(\mathrm{row}(C) \geq k)$*. Moreover, for almost any* $F \in \mathcal{R}^{r\times k}$ $(G \in \mathcal{R}^{k\times m})$*,* (E, A, F, G) *is R-controllable (R-observable), where* $k = \mathrm{cyc}(E, A)$*.*

(b) it is impulse controllable (impulse observable) only if $\mathrm{col}(B) \geq k$ $(\mathrm{row}(C) \geq k)$*. Moreover, for almost any* $F \in \mathcal{R}^{r\times k}$ $(G \in \mathcal{R}^{k\times m})$*,* (E, A, F, G) *is impulse controllable (impulse observable), where* $k = \mathrm{cyc}_\infty(E, A)$*.*

(c) it is strongly controllable (strongly observable) only if $\mathrm{col}(B) \geq k$ *(*$\mathrm{row}(C) \geq k$*). Moreover, for almost any* $F \in \mathcal{R}^{r\times k}$ *(*$G \in \mathcal{R}^{k\times m}$*),* (E, A, F, G) *is strongly controllable (strongly observable), where* $k = \mathrm{Cyc}(E, A)$*.*

Theorem 5 *Assume that the system (1) is regularizable by output feedback, then,*

(a) for almost any $K \in R^{r\times m}$*,* $F \in R^{r\times k}$*, and* $G \in R^{k\times m}$*,* $(E, A + BKC, F, G)$ *is R-controllable and R-observable, where* $k = cyc(E, A, B, C)$*.*

(b) *for almost any* $K \in \mathcal{R}^{r\times m}$*,* $F \in \mathcal{R}^{r\times k}$*, and* $G \in \mathcal{R}^{k\times m}$*,* $(E, A + BKC, F, G)$ *is impulse controllable and impulse observable, where* $k = \mathrm{cyc}_\infty(E, A, B, C)$*.*

(c) *for almost any* $K \in \mathcal{R}^{r\times m}$*,* $F \in \mathcal{R}^{r\times k}$*, and* $G \in \mathcal{R}^{k\times m}$*,* $(E, A + BKC, F, G)$ *is strongly controllable and strongly observable, where* $k = \mathrm{Cyc}(E, A, B, C)$*.*

6 Conclusion

This paper studies structural properties of both finite poles and the infinite pole of LTI singular systems under output feedback. The ability of output feedback to change the structure of the finite and infinite pole of the system is characterized through some new concepts regarding the multiplicity of the pole or the indices of the system. The determination of these multiplicities and indices are discussed. The number of the infinite pole that can be eliminated by output feedback is given. An assignability equivalence is established between the variable finite poles and the poles of a controllable and observable non-singular system. Consequently, all the finite poles, except the fixed ones (corresponding to the uncontrollable or unobservable modes), can be separated from each other and shifted away from any given finite set. It is also shown that the minimal number of inputs (outputs) required for controllability (observability) is equal to the output feedback cycle indices of the singular system.

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