Closed-Loop Structure of Discrete–Time H^∞ Controller

Waree Kongprawechnon[1], Shun Ushida[2], Hidenori Kimura[2]

Abstract

This paper is concerned with the investigation of the closed-loop structure of a discrete–time H^∞ control system. It is shown that discrete–time H^∞ controller is represented, like Linear Quadratic Gaussian control, as a pseudo state feedback, that is, a state feedback interconnected with an observer. However, in the discrete-time H^∞ control problem the control structure is more complicated since we cannot choose the state feedback and the observer independently.

1 Introduction

H^∞ control theory has received much attention over the last two decades(see Francis[4], Doyle et al[3], Stoorvogel[9], Mirkin[8] and the references therein). Early results for the H^∞ control problem were derived for the continuous–time case. However, in practical applications controllers operate mainly in discrete–time. Furthermore, we can also use a discrete–time controller to control a continuous–time system. An approach is discretizing the system first and then using H^∞ control designed for discrete–time systems. There are many results in this direction.(e.g. Chen et al.[1], Bamieh et al.[2] and Yamamoto[11]). Also, certain systems are in themselves inherently discrete, and certainly for these systems it is useful to have results available for discrete–time H^∞ control problem.

In a previous paper[7] , we studied H^∞ control for discrete–time systems. We have obtained a necessary and sufficient condition under which an H^∞ norm bound can be achieved by an internally stabilizing output feedback controller(Normalized version). Note that to derive the unnormalized version we can solve it by using the scaled plant Like in continuous–time case. In this paper, we investigate the structure of H^∞ controller in details and show its intrinsic pseudo–state feedback structure. Also, we derive a set of necessary and sufficient conditions for the existence of strictly proper H^∞ controllers. This problem has been studied before in Stoorvogel[10] and Mirkin[8]. However, by using the chain–scattering approach, our derivation is much simpler and it clarifies the controller structure in a straightforward way. In this paper, we use the following notations.

$$J_{mr} := \begin{bmatrix} I_m & 0 \\ 0 & -I_r \end{bmatrix}, \quad J = J_{mr}, \quad J' = J_{pq}, \quad J'' = J_{mq}$$

[1]Sirindhorn International Institute of Technology Thammasat University, Patum–thani, Thailand
[2]Faculty of Engineering, The University of Tokyo, Bunkyo, Tokyo, Japan

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] := C(zI-A)^{-1}B + D$$

$\sigma(A)$ is the maximum spectral radius of constant matrix A, $\mathbb{R}_{m\times r}$ is the set of real $m\times r$ matrices, $\mathbb{RL}^{\infty}_{m\times r}$ is the set of all $m\times r$ rational matrices without pole on the unit circle, $\mathbb{RH}^{\infty}_{m\times r}$ is the set of all $m\times r$ rational stable proper matrices. $\mathbb{BH}^{\infty}_{m\times r}$ is the subset of $\mathbb{RH}^{\infty}_{m\times r}$ whose norm is less than 1.

2 Preliminaries and Problem Formulation

2.1 Plant

We consider a Linear time–invariant discrete–time system described by

$$x_{k+1} = Ax_k + B_1 w_k + B_2 u_k, \tag{2.1a}$$

$$z_k = C_1 x_k + D_{11} w_k + D_{12} u_k, \tag{2.1b}$$

$$y_k = C_2 x_k + D_{21} w_k, \tag{2.1c}$$

where z is the controlled error ($\dim(z) = m$), y is the observation output ($\dim(y) = q$), w is the exogeneous input ($\dim(w) = r$), u is the control input ($\dim(u) = p$). We make the usual assumptions that

(A1) (A, B_2) is stabilizable and (A, C_2) is detectable.

(A2) rank $D_{21} = q$, rank $D_{12} = p$.

2.2 Standard H^{∞} Control Problem

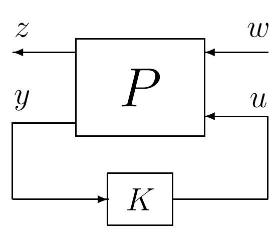

Fig. 1 H^{∞} Control Scheme

The plant (2.1) can be written in the input/output form as

$$\begin{bmatrix} z(z) \\ y(z) \end{bmatrix} = \begin{bmatrix} P_{11}(z) & P_{12}(z) \\ P_{21}(z) & P_{22}(z) \end{bmatrix} \begin{bmatrix} w(z) \\ u(z) \end{bmatrix}. \tag{2.2}$$

A feedback control law

$$u(z) = K(z)y(z) \tag{2.3}$$

generates the closed-loop transfer function $\Phi(z)$ from $w(z)$ to $z(z)$ given by

$$\Phi := LF(P; K) := P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21}. \tag{2.4}$$

The objective is to find a control law (2.3) which internally stabilizes the closed-loop system of Fig. 1, achieving the normalized norm bound of $\Phi(z)$, that is,

$$\| \Phi \|_\infty < 1. \tag{2.5}$$

2.3 Chain–Scattering Representation

Assuming that P_{21} is invertible, we have $w = P_{21}^{-1}(y - P_{22}u)$. Substituting this relation in the first equation of (2.2) yields $z = (P_{12} - P_{11}P_{21}^{-1}P_{22})u + P_{11}P_{21}^{-1}y$. Therefore, if we write

$$G := CHAIN(P) := \begin{bmatrix} P_{12} - P_{11}P_{21}^{-1}P_{22} & P_{11}P_{21}^{-1} \\ -P_{21}^{-1}P_{22} & P_{21}^{-1} \end{bmatrix},$$

the relation (2.2) is alternatively represented as

$$\begin{bmatrix} z(z) \\ w(z) \end{bmatrix} = \begin{bmatrix} G_{11}(z) & G_{12}(z) \\ G_{21}(z) & G_{22}(z) \end{bmatrix} \begin{bmatrix} u(z) \\ y(z) \end{bmatrix}. \tag{2.6}$$

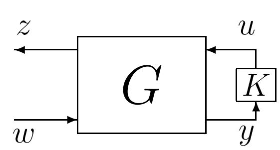

Fig. 2 Chain–Scattering Representation of the system

A feedback control law (2.3) applied to the chain–scattering representation of plant (2.6) generates the closed–loop transfer function $\Phi(z)$ given by

$$\Phi(z) := HM(G; K) := (G_{11}K + G_{12})(G_{21}K + G_{22})^{-1}. \tag{2.7}$$

The symbol HM stands for the *HoMographic Transformation.*[6] The properties of the transformation HM are listed up in the following lemmas, which are based on the work of Kimura.[5]. Their proof is essentially the same as in the continuous–time case.

Lemma 2.1. *Properties of* HM
(*i*) *If* P_{21}^{-1} *exists,* $LF(P; K) = HM(CHAIN(P); K)$.
(*ii*) $HM(I; K) = K$.
(*iii*) $HM(G_1, HM(G_2; K)) = HM(G_1G_2; K)$.
(*iv*) *If* G^{-1} *exists,* $HM(G; K) = F$ *implies* $K = HM(G^{-1}; F)$.

Next, we recall the following theorem from Kongprawechnon and Kimura.[7].

Theorem 2.2. *Under the assumptions (A1) and (A2), the normalized H^∞ control problem is solvable iff*

(i) there exists a solution $X \geq 0$ of the algebraic Riccati equation

$$X = A^T X A + C_1^T C_1 - F^T (D_z^T J D_z + B^T X B) F \tag{2.8}$$

such that $\hat{A}_G := A + BF$ is stable,

(ii) there exists a solution $Y \geq 0$ of the algebraic Riccati equation

$$Y = AYA^T + B_1 B_1^T + L(D_w J D_w^T - CYC^T) L^T \tag{2.9}$$

such that $\hat{A}_H := A + LC$ is stable,

(iii) $\sigma(XY) < 1$,

(iv) there exists a nonsingular matrix E_z such that $D_z^T J D_z + B^T X B = E_z^T J' E_z$ holds,

(v) there exists a nonsingular matrix E_w such that $D_w J D_w^T - CYC^T = E_w J'' E_w^T$ holds, where

$$\begin{aligned}
F &:= \begin{bmatrix} F_w \\ F_u \end{bmatrix} = -(D_z^T J D_z + B^T X B)^{-1}(D_r^T C_1 + B^T X A), \\
L &:= \begin{bmatrix} L_z & L_y \end{bmatrix} = -(B_1 D_c^T + AYC^T)(D_w J D_w^T - CYC^T)^{-1}, \\
D_r &:= \begin{bmatrix} D_{11} & D_{12} \end{bmatrix}, \ D_c := \begin{bmatrix} D_{11} \\ D_{21} \end{bmatrix}, \ D_w := \begin{bmatrix} -I & D_{11} \\ 0 & D_{21} \end{bmatrix}, \ D_z := \begin{bmatrix} D_{11} & D_{12} \\ I & 0 \end{bmatrix}, \\
D_u &= \begin{bmatrix} -D_{12} & 0 \\ 0 & I \end{bmatrix}, \ C := \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}, B := \begin{bmatrix} B_1 & B_2 \end{bmatrix}, \ B_u := \begin{bmatrix} B_2 & 0 \end{bmatrix}.
\end{aligned}$$

In that case, a desired controller is given by

$$K = HM(\Pi_{11}^{-1}; S), \tag{2.10}$$

where

$$\begin{aligned}
\Pi_{11}^{-1} &= QV_w^{-1}, \qquad U := (I - YX)^{-1}. \\
Q &:= \left[\begin{array}{c|cc} A + B_1 F_w + B_2 F_u & U \begin{bmatrix} B_2 + L_z D_{12} & -L_y \end{bmatrix} & \\ \hline F_u & I & 0 \\ C_2 + D_{21} F_w & 0 & I \end{array}\right],
\end{aligned} \tag{2.11}$$

and V_w is a nonsingular matrix satisfying

$$V_w^T J_{pq} V_w = (B_u + LD_u)^T X (I - YX)^{-1} (B_u + LD_u) + D_u^T (D_w J D_w^T - CYC^T)^{-1} D_u \tag{2.12}$$

and S is an arbitrary matrix in $\mathbb{BH}^\infty$.

Note That: We can derive the unormalized version of Theorem 2.2 by using the scaled plant; that is

$$P_\gamma := \begin{bmatrix} \gamma^{-1}P_{11} & P_{12} \\ \gamma^{-1}P_{21} & P_{22} \end{bmatrix} = \left[\begin{array}{c|cc} A & B_1 & \gamma B_2 \\ \hline \gamma^{-1}C_1 & \gamma^{-1}D_{11} & D_{12} \\ \gamma^{-1}C_2 & \gamma^{-1}D_{21} & 0 \end{array}\right],$$

where γ is a positive number.

3 Main Results

In this section, we will consider the closed–loop structure of an H^∞ controller. From equation (2.10), (2.11) and the cascade property of HM, we have

$$K = HM(QV_w^{-1}; S) = HM(Q; HM(V_w^{-1}; S)). \tag{3.1}$$

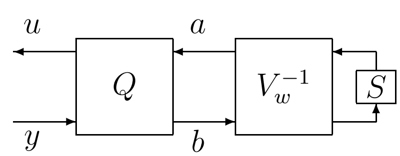

Fig. 3 Chain–Scattering Representation of the Controller

Q given in (2.11) is described in the state–space as

$$\begin{aligned} \xi_{k+1} &= (A + B_1F_w + B_2F_u)\xi_k + U(B_2 + L_zD_{12})a_k - UL_yb_k, & (3.2a) \\ u_k &= F_u\xi_k + a_k, & (3.2b) \\ y_k &= (C_2 + D_{21}F_w)\xi_k + b_k, & (3.2c) \end{aligned}$$

where ξ_k is the state of the controller. The controller (3.1) is obtained by introducing the relation $a_k = HM(V_w^{-1}; S)b_k$. Hence in continuous–time case the signal a does not depend of the solution of algebraic Riccati equation while the discrete–time case does. The controller can be rewritten as

$$\begin{aligned} \xi_{k+1} &= (A+B_1F_w+B_2F_u)\xi_k + U\left((B_2+L_zD_{12})HM(V_w^{-1}; S) - L_y\right)b_k, & (3.3a) \\ u_k &= F_u\xi_k + HM(V_w^{-1}; S)b_k, & (3.3b) \\ b_k &= y_k - (C_2 + D_{21}F_w)\xi_k. & (3.3c) \end{aligned}$$

The block-diagram of the controller is illustrated in Fig. 4. The block diagram is similar to the continuous–time case.

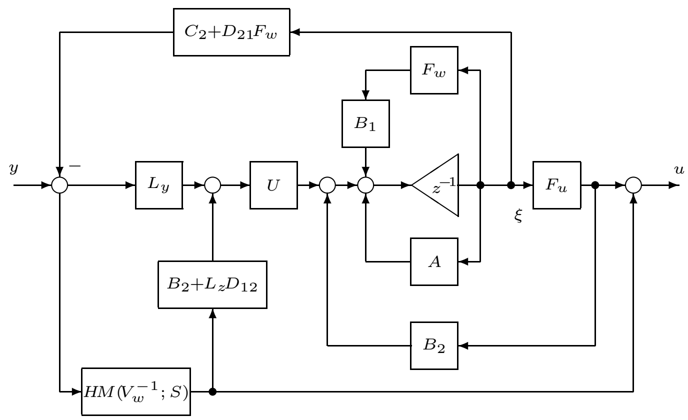

Fig. 4 Block Diagram of the H^{∞} Controller

First, we will consider under the constraint imposed upon the controller to be strictly proper. As was shown by Mirkin et al[8], sampled–data control problems can always be formulated as discrete–time problems with strictly causal controllers. Hence, the consideration of strictly proper controllers does not lead to any loss of generality in most cases. To parameterize all strictly proper controllers is to extract the set of all strictly proper controllers from $K = HM(Q; HM(V_w^{-1}; S))$. In other words, one should find whether there exists a transfer matrix $S \in \mathbb{BH}^{\infty}_{(p+q)\times(p+q)}$ such that $K(\infty) = HM(Q; HM(V_w^{-1}; S))(\infty) = 0$. To this end, note that $Q(\infty) = I$. Due to (ii) of Lemma 2.1, $K(\infty) = HM(V_w^{-1}; S)(\infty)$. Let $V_w = \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix}$. To make $K(\infty) = HM(V_w^{-1}; S)(\infty) = 0$, we should choose $S = HM(V_w; 0) = V_{12}V_{22}^{-1}$, from (iv) of Lemma 2.1. Since $S \in \mathbb{BH}^{\infty}_{(p+q)\times(p+q)}$, hence

$$\|V_{12}V_{22}^{-1}\| < 1. \tag{3.4}$$

Thus to adjust Theorem 2.2 to the case of strictly proper controller, one has to add (3.4) to the conditions of Theorem 2.2. From Lemma 2.1 of Ionescu et al[13], V_w can be chosen block lower–left triangular, that is, $V_w = \begin{bmatrix} V_{11} & 0 \\ V_{21} & V_{22} \end{bmatrix}$, which always satisfy (3.4).

Next we will consider the structure of the central controller. From $K = HM(Q; HM(V_w^{-1}; S)) = HM(Q; R)$, where $R := HM(V_w^{-1}; S)$, we have $S = HM(V_w; R) = V_{11}R(V_{21}R + V_{22})^{-1}$. If we choose $S = 0$, then we have $R = 0$. Moreover, we obtain the so–called central controller, which is described as

$$\xi_{k+1} = A\xi_k + B_1\hat{w}_{0k} + B_2u_k - UL_y(y_k - C_2\xi_k - D_{21}\hat{w}_{0k}), \tag{3.5a}$$

$$u_k = F_u\xi_k, \tag{3.5b}$$

$$\hat{w}_{0k} = F_w\xi_k. \tag{3.5c}$$

The representation (3.5a)-(3.5c) clarifies the observer structure of the central controller. Fig. 5 illustrates the block diagram of this controller.

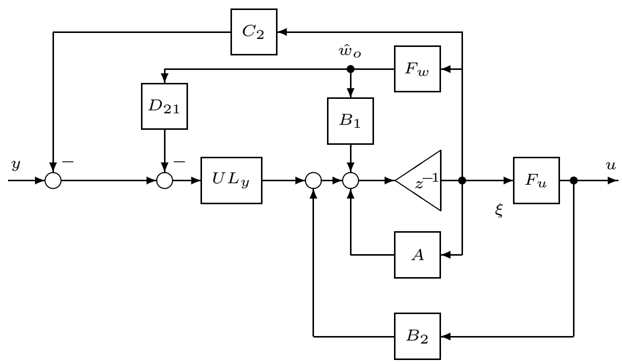

Fig. 5 Block Diagram of the Central Controller

Like Stroovogel[9] and Green et al[14], we can see that this representation clarified the pseudo–state feedback structure of the central controller.

4 Examples

In this section, a collection of simple examples is given. in order to get an idea of the structure of H^∞control.

Example 4.1 This example is the so-called two-block case. Numerical computations are performed on the control design software MATLAB. We consider the following second-order system

$$
\begin{aligned}
x_{k+1} &= \begin{bmatrix} 0.1 & 0 \\ 0 & 2 \end{bmatrix} x_k + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_k, \\
z_k &= \begin{bmatrix} 0 & 1.7 \end{bmatrix} x_k + u_k, \\
y_k &= \begin{bmatrix} 0 & 4.2102 \end{bmatrix} x_k + \begin{bmatrix} 0 & 0.7988 \end{bmatrix} w_k.
\end{aligned}
$$

The solution of (2.8) and (2.9) are given respectively by

$$X = 0, \qquad Y = \begin{bmatrix} 0 & 0 \\ 0 & 0.1205 \end{bmatrix} > 0.$$

The matrices $\hat{A}_G$ and $\hat{A}_H$ are given respectively by

$$\hat{A}_G = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.3 \end{bmatrix}, \qquad \hat{A}_H = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.5 \end{bmatrix},$$

which satisfy the conditions of Theorem 2.2. In this case, the central controller is given by

$$\begin{aligned} \xi_{k+1} &= \begin{bmatrix} 0.1 & 0 \\ 0 & 0.3 \end{bmatrix} \xi_k + \begin{bmatrix} 0 \\ 0.1801 \end{bmatrix} \nu_k, \\ u_k &= \begin{bmatrix} 0 & -1.7 \end{bmatrix} \xi_k - 0.1973\nu_k, \\ \nu_k &= y_k - \begin{bmatrix} 0 & 4.2102 \end{bmatrix} \xi_k. \end{aligned}$$

◊

Remark: This is the case that $A - B_2 D_{12}^{-1} C_1$ is stable. Therefore, Condition (i) of Theorem 2.2 is unnecessary. Also, Condition (iii) holds automatically and Condition (iv) can be checked easily. ♡

Example 4.2 We consider another second-order system. This is an example of four-block case.

$$\begin{aligned} x_{k+1} &= \begin{bmatrix} 0 & 0.9665 \\ 1.1387 & 0 \end{bmatrix} x_k + \begin{bmatrix} 0 & 0 \\ 1.2828 & 0 \end{bmatrix} w_k + \begin{bmatrix} 0 \\ 6.0231 \end{bmatrix} u_k, \\ z_k &= \begin{bmatrix} 0.1884 & 0 \\ 0 & 0 \end{bmatrix} x_k + \begin{bmatrix} 0 \\ 0.7821 \end{bmatrix} u_k, \\ y_k &= \begin{bmatrix} 3.3510 & 0 \end{bmatrix} x_k + \begin{bmatrix} 0 & 5.4403 \end{bmatrix} w_k. \end{aligned}$$

Then we obtain

$$X = \begin{bmatrix} 0.0521 & 0 \\ 0 & 0.0486 \end{bmatrix} > 0$$

and

$$Y = \begin{bmatrix} 3.4473 & 0 \\ 0 & 3.6908 \end{bmatrix} > 0.$$

where satisfy all conditions of Theorem 2.2. In this case, the central controller is given by

$$\begin{aligned} \xi_{k+1} &= \begin{bmatrix} 0 & 0.9665 \\ 0.2993 & 0 \end{bmatrix} \xi_k - \begin{bmatrix} 0 \\ 0.0549 \end{bmatrix} b_k, \\ u_k &= \begin{bmatrix} -0.1433 & 0 \end{bmatrix} \xi_k - 0.0263 b_k, \\ b_k &= y_k - \begin{bmatrix} 3.3510 & 0 \end{bmatrix} \xi_k. \end{aligned}$$

◊

5 Conclusion

The closed–loop structure of discrete–time H^∞ control has been discussed. The existence condition for a strictly proper H^∞ controller for discrete–time systems has also been derived. We believe that the result derived in this paper may be a useful tool in solving various control problems with the H^∞ performance measure.

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