Equivalence of Finite Pole Assignability of LTI Singular Systems by Output Feedback

Runyi Yu
Dept. of Electrical and Electronic Eng.
Eastern Mediterranean University
Gazimagusa, Mersin 10, Turkey
runyi.yu@emu.edu.tr

Dianhui Wang
Dept. of Computer Sci. and Computer Eng.
La Trobe University
Bundoora, VIC 3083, Australia
dhwang@cs.latrobe.edu.au

Abstract

This paper shows that the assignability of finite poles of a strongly controllable and observable singular system (E, A, B, C) is equivalent to the pole assignability of a non-singular system (A_s, B_s, C_s) of order rank(E). Consequently, all the existing results on pole assignment of non-singular systems can be extended to singular systems as far as the finite poles are concerned.

1 Introduction

Consider the linear time-invariant singular system described by

$$E\dot{x} = Ax + Bu, \; y = Cx \tag{1}$$

where $x \in \mathcal{R}^n$ is the state of the system, $u \in \mathcal{R}^r$ and $y \in \mathcal{R}^m$ are the input and output vector of the system respectively, and E, A, B, C are real constant matrices of appropriate sizes. E is assumed to be singular with $0 <$rank$(E)$$= q < n$.

If we apply a static output feedback

$$u = Ky + v \tag{2}$$

the closed-loop system (1) becomes

$$E\dot{x} = (A + BKC)x + Bu, \; y = Cx \tag{3}$$

It is well known that a singular system has a complicated structure and may contain infinite pole (impulse mode) in addition to the finite poles. It is desirable to eliminate the infinite poles as well as to assign the finite poles. It is shown [7] that if the system (1) is impulse controllable and observable, then its infinite pole can be completely eliminated by output feedback. [1] shows that the finite poles can be freely assigned and the infinite poles can all be eliminated by state feedback if the system is strongly controllable. In this paper, we shall study the assignability of the finite poles of the closed-loop system (3) by output feedback. A so-called equivalence of pole assignability is established between singular systems and non-singular systems. Specifically, we show that the finite poles of the closed-loop system (3) can be assigned as much as the finite poles of a controllable and observable system (A_s, B_s, C_s) of order rank(E) if the system (1) is strongly controllable and observable. Consequently, the existing result regarding pole assignability under output feedback for non-singular systems can be directly extended to singular systems (E, A, B, C) as far as the finite poles are concerned.

2 Preliminaries

Let $x \in \mathcal{R}^n$, and $f(x)$ denote a polynomial in the ring of polynomials $\mathcal{R}[x]$. A property is called generic [5] if there exits a polynomial $f(x) \in \mathcal{R}[x]$ not identically zero such that the property holds for any $x \in \mathcal{S} = \mathcal{R}^n - \mathcal{N}(f)$, where $\mathcal{N}(f) = \{x \mid f(x) = 0, x \in \mathcal{R}^n\}$. In other words, a generic property holds for almost any $x \in \mathcal{R}^n$.

Lemma 1 *[2] The following statement are true.*

1. *The system* (1) *is impulse controllable (impulse observable) if and only if* $\text{rank}\begin{bmatrix} E & 0 & 0 \\ A & E & B \end{bmatrix} = n + \text{rank}(E)$ *(*$\text{rank}\begin{bmatrix} E & A \\ 0 & E \\ 0 & C \end{bmatrix} = n + \text{rank}(E)$*).*

2. *The system* (1) *is strongly controllable (strongly observable) if and only if it is impulse controllable (impulse observable) and* $\text{rank}\begin{bmatrix} sE - A & B \end{bmatrix} = n$ *(*$\text{rank}\begin{bmatrix} sE - A \\ C \end{bmatrix} = n$*) for any* $s \in \mathcal{C}$*.*

Lemma 2 *Let* $A \in \mathcal{R}^{n\times l}$*,* $B \in \mathcal{R}^{n\times r}$*, and* $C \in \mathcal{R}^{m\times l}$ *be constant matrices, let* $K \in \mathcal{R}^{r\times m}$ *be a variable matrix, then,*

$$\text{g.r.}\{\text{rank}(A + BKC), K \in \mathcal{R}^{r\times m}\} = \min\{\text{rank}\begin{bmatrix} A & B \end{bmatrix},\ \text{rank}\begin{bmatrix} A \\ C \end{bmatrix}\}$$

where g.r. *stands for generic rank, i.e., the maximum rank of the matrix of* $A + BKC$ *as* K *varies in* $\mathcal{R}^{r\times m}$*. Furthermore, for almost any* $K \in \mathcal{R}^{r\times m}$*,*

$$\text{g.r.}\{\text{rank}(A + BKC), K \in \mathcal{R}^{r\times m}\} = \text{rank}(A + BKC)$$

Proof: The first part is available in [6]. Let us prove the second part.

Denote $r^* = \min\left\{\text{rank}\begin{bmatrix} A & B \end{bmatrix},\ \text{rank}\begin{bmatrix} A \\ C \end{bmatrix}\right\}$. We consider a polynomial $f(k)$ defined as the sum of the squares of all possible minors of order r^* of the matrix $A + BKC$, where the vector k is composed of all the elements of the matrix K. Clearly the set of zeros of $f(k)$ is the complement of the set whose element achieves the maximum rank, and $f(k)$ is nonzero due to the first part. Hence, the result follows.

The second part of the lemma is first presented in a recent paper [4], where singular value decomposition of the matrices is used in the proof. It is pointed out that the proof given above is straightforward and much simpler than that in [4].

3 Main Result

Let us now study the assignability of the finite poles of the system (1). To this end, we assume that the singular system (1) is both strongly controllable and observable. In other words, both the exponential modes and the impulse modes are assumed to be controllable and observable. The strong controllability is assumed, because it is required even for the state feedback to make the closed-loop system impulse-free and assign the finite pole arbitrarily [1].

We recall [2] that two restricted equivalent system have the same poles (both finite and the infinite). Moreover, both controllability and observability are invariant under restricted equivalent transform and output feedback.

Let Q_1, P_1 be non-singular matrices satisfying $Q_1 E P_1 = \text{diag}(I_q, 0)$. The system (1) is restricted equivalent to

$$\overline{E}\dot{\overline{x}} = \overline{A}\overline{x} + \overline{B}u, \; y = \overline{C}\overline{x} \tag{4}$$

where the new state $\overline{x} = P_1^{-1}x$, and

$$\overline{E} = Q_1 E P_1, \; \overline{A} = Q_1 A P_1 = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \; \overline{B} = Q_1 B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \overline{C} = C P_1 = \begin{bmatrix} C_1 & C_2 \end{bmatrix}$$

Following Lemma 1, the system (1) is impulse controllable if and only if $\text{rank}\begin{bmatrix} A_{22} & B_2 \end{bmatrix} = n - q$; the system (1) is impulse observable if and only if $\text{rank}\begin{bmatrix} A_{22} \\ C_2 \end{bmatrix} = n - q$. It follows from Lemma 2 that $A_{22} + B_2\overline{K}C_2$ is invertible for almost any $K \in \mathcal{R}^{r\times m}$. For such a matrix K, let the output feedback

$$u = \overline{K}y + v$$

The system (4) has the following closed-loop form

$$\overline{E}\dot{\overline{x}} = \begin{bmatrix} A_{11} + B_1\overline{K}C_1 & A_{12} + B_1\overline{K}C_2 \\ A_{21} + B_2\overline{K}C_1 & A_{22} + B_2\overline{K}C_2 \end{bmatrix} \overline{x} + \overline{B}v, \; y = \overline{C}\overline{x} \tag{5}$$

Define two non-singular matrices

$$Q_2 = \begin{bmatrix} I_q & -X_{12}X_{22}^{-1} \\ 0 & I_{n-q} \end{bmatrix}, P_2 = \begin{bmatrix} I_q & 0 \\ -X_{22}^{-1}X_{21} & X_{22}^{-1} \end{bmatrix}$$

where $X_{ij} = A_{ij} + B_i\overline{K}C_j, \; i,j = 1,2$. And make the state transformation $\widetilde{x} = P_2^{-1}\overline{x}$. The system (5) is restricted equivalent to

$$\widetilde{E}\dot{\widetilde{x}} = \widetilde{A}\widetilde{x} + \widetilde{B}v, \; y = \widetilde{C}\widetilde{x} \tag{6}$$

where

$$\widetilde{E} = \overline{E}, \; \widetilde{A} = Q_2\overline{A}P_2 = \begin{bmatrix} A_s & 0 \\ 0 & I_{n-q} \end{bmatrix}$$

$$\widetilde{B} = Q_2\overline{B}P_2 = \begin{bmatrix} B_s \\ B_2 \end{bmatrix}, \; \widetilde{C} = Q_2\overline{C}P_2 = \begin{bmatrix} C_s & C_2X_{22}^{-1} \end{bmatrix}$$

where $A_s = X_{11} - X_{12}X_{22}^{-1}X_{21}$, $B_s = B - X_{12}X_{22}^{-1}B_2$, and $C_s = C_1 - C_2X_{22}^{-1}X_{21}$. Since both controllability and observability are invariant under restricted equivalent transform and output feedback, we can easily obtain

Lemma 3 *Assume that the system (1) is impulse controllable and observable. Then it is strongly controllable if and only if the pair (A_s, B_s) is completely controllable; It is strongly observable if and only if the pair (As, C_s) is observable.*

Up to now, we have obtained a restricted equivalent decomposition of the closed-loop system under output feedback. We summarize this decomposition in the next lemma.

Lemma 4 *If the system (1) is strongly controllable and observable, then for almost any output feedback (2), the closed-loop system (3) is restricted equivalent to a system with the following decomposition*

$$\begin{bmatrix} I_q & 0 \\ 0 & 0 \end{bmatrix} \dot{\widetilde{x}} = \begin{bmatrix} A_s & 0 \\ 0 & I_{n-q} \end{bmatrix} \widetilde{x} + \begin{bmatrix} B_s \\ B_f \end{bmatrix} v, \; y = \begin{bmatrix} C_s & C_f \end{bmatrix} \widetilde{x}$$

Furthermore, the subsystem (A_s, B_s, C_s) of order q is both completely controllable and observable.

Apply output feedback

$$v = \widetilde{K} y + w$$

to the system (6), we get the characteristic equation of the resultant closed-loop system

$$d(s) = \det \begin{bmatrix} sI - A_s - B_s\widetilde{K}C_s & -B_s\widetilde{K}C_f \\ -B_f\widetilde{K}C_s & -I - B_f\widetilde{K}C_f \end{bmatrix} = 0$$

Note that $I + B_f\widetilde{K}C_f$ is invertible for almost any $\widetilde{K} \in \mathcal{R}^{r\times m}$, we have

$$\begin{aligned} d(s) &= \det(-I - B_f\widetilde{K}C_f) \det[sI - A_s - B_s\widetilde{K}C_s + B_s\widetilde{K}C_f(I + B_f\widetilde{K}C_f)^{-1}B_f\widetilde{K}C_s] \\ &= \det(-I - B_f\widetilde{K}C_f) \det\left[sI - A_s - B_s[\widetilde{K} - \widetilde{K}C_f(I + B_f\widetilde{K}C_f)^{-1}B_f\widetilde{K}]C_s\right] \end{aligned}$$

In order to recover the output feedback matrix in the original input space, the following lemma is needed.

Lemma 5 *If both $I + B_f\widetilde{K}C_f$ and $I - K_sC_fB_f$ are invertible, then the following two matrix equations are equivalent*

$$K_s = \widetilde{K} - \widetilde{K}C_f(I + B_f\widetilde{K}C_f)^{-1}B_f\widetilde{K}$$
$$\widetilde{K} = (I - K_sC_fB_f)^{-1}K$$

Proof. The equivalence can be proved by direct matrix manipulations using the matrix equations $Y(I + XY)^{-1} = (I + YX)^{-1}Y$ and $X(I + X)^{-1} = I - (I + X)^{-1}$.

We are now ready to present the following result regarding the finite pole assignability of singular systems.

Theorem 1 *Assume that the singular system (1) is strongly controllable and observable. The assignability of the finite poles of the singular system (1) by output feedback is equivalent to that of the poles of a non-singular system (A_s, B_s, C_s) of order q that is completely controllable and observable.*

Proof. Using the matrices introduced above, $I - K_sC_fB_f$ is invertible for almost any $K_s \in \mathcal{R}^{r\times m}$. Let $\widetilde{K} = (I - K_sC_fB_f)^{-1}K_s$. From Lemma 5, under the invertibility constraints, the set of the finite poles

$$\begin{aligned} \sigma(A_s + B_sK_sC_s) &= \sigma(\widetilde{E}, \widetilde{A} + \widetilde{B}\widetilde{K}\widetilde{C}) = \sigma(\overline{E}, \overline{A} + \overline{BKC} + \overline{B}\widetilde{K}\overline{C}) \\ &= \sigma(\overline{E}, \overline{A} + \overline{B}(\overline{K} + \widetilde{K})\overline{C}) = \sigma(E, A + BKC) \end{aligned}$$

where $K = \overline{K} + \widetilde{K}$. The assignability equivalence then follows from the generic property of the matrices $\widetilde{K}$ and K_s that yield the same desired poles. This completes the proof.

The above theorem implies that under the assumption of strong controllability and observability what we can say about the assignment of the finite poles of the closed-loop singular system is as much as what we understand about the assignment of the poles of a controllable and observable triple (A_s, B_s, C_s) of order rank(E). We call this *the equivalence of finite pole assignability under output feedback*.

Following the above equivalence, all the existing results on output feedback for non-singular systems can be extended straightforward to singular systems. We now give an extension of a result in [3] regarding the separation and shifting of the finite poles.

Theorem 2 *Assume that the singular system (1) is strongly controllable and observable. For almost any output feedback (2), the closed-loop system (3) has rank(E) distinct finite poles, and they are away from any given finite set in the complex plane.*

4 Conclusion

This paper studies the finite pole assignment problem of LTI singular systems by output feedback. It first presents a restricted equivalent decomposition for the closed-loop system under the assumption of strong controllability and observability. This decomposition allows the establishment of an equivalence of pole assignability between the LTI singular system and a non-singular systems of low order. Consequently, so far as the finite poles are concerned, the existing results regarding static output feedback for non-singular systems can be directly extended to singular systems.

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