Multivariable Extremum Seeking Feedback: Analysis and Design[1]

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Abstract

The paper provides a multivariable extremum seeking scheme, the first for systems with general time-varying parameters. We derive a stability test in a simple SISO format and develop a systematic design algorithm based on standard LTI control techniques to satisfy the stability test. We also supply an analytical quantification of the level of design difficulty in terms of the number of parameters and in terms of the shape of the unknown equilibrium map. Moreover, we remove the requirement of slow forcing for plants with strictly proper output dynamics (and consequent slow convergence) present in earlier works.

1 Introduction

Extremum seeking, an adaptive control technique with several successful applications (See [3] and references therein) has witnessed a resurgence of interest after the publication of the first stability studies in [3] and [2]. Though we now have the means to check for stability of these schemes [2, 4, 6], the need for rigorous design guidelines guaranteeing performance has been strongly felt [2, 4].

The first stability analysis of extremum seeking for a general nonlinear plant was developed in [3]. In [2], dynamic compensation was proposed for providing stability guarantees and fast tracking of changes in plant operating conditions for single parameter extremum seeking. However, the analysis is valid only for step changes in plant parameters and the result led to a very difficult design problem.

Rotea [4] and Walsh [6] provided the first studies of multivariable extremum seeking schemes. Their results were for plants with constant parameters. Furthermore, for strictly proper output dynamics, their stability criteria would require use of slow forcing and consequent slow convergence, a limitation inherited from the analysis method introduced in [3]. A systematic design procedure is absent in both [2, 3] and [4, 6].

This paper solves an array of problems remaining after [2, 3, 4, 6]:

1. Provides the first multivariable extremum seeking scheme for general time-varying parameters by applying averaging on a system in a form different than [2].

2. Derives a stability test in a simple SISO format.

3. Develops a systematic design algorithm based on standard LTI control techniques to satisfy the stability test.

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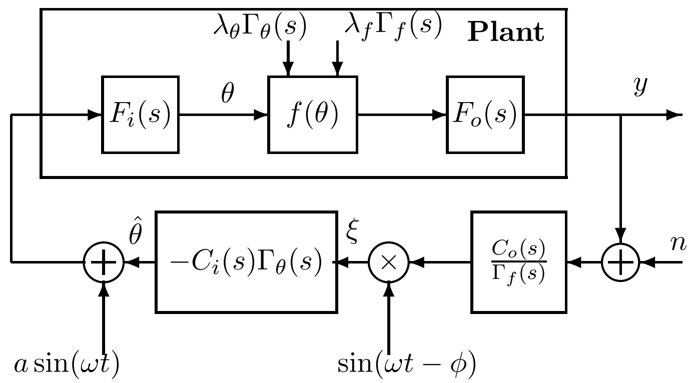

Figure 1: Extension of the extremum seeking algorithm to non-step changes in θ^* and f^*

4. Supplies an analytical quantification of the level of design difficulty in terms of the number of parameters and in terms of the shape of the unknown equilibrium map.

5. Removes the requirement of slow forcing for plants with strictly proper output dynamics (and consequent slow convergence) present in earlier works. This is achieved by using the modulation properties of the Laplace transform.

For ease of understanding of the analysis, design and implementation, and also for its importance, the single parameter case is presented first. Section 2 provides the problem formulation, Subsection 2.1 the stability analysis and Subsection 2.2 the design algorithm. Section 4 presents the analysis and Section 5 the design for a general multivariable scheme.

2 Output Extremization in Single Parameter Extremum Seeking

Figure 1 shows the nonlinear plant with linear dynamics along with the extremum seeking loop. We let $f(\theta)$ be a function of the form:

$$f(\theta) = f^*(t) + \frac{f''}{2}\left(\theta - \theta^*(t)\right)^2, \tag{2.1}$$

where $f'' > 0$ is constant but unknown. Any function $f(\theta)$ that has a quadratic minimum at θ^* can be approximated locally by Eqn. (2.1). The assumption $f'' > 0$ is made without loss of generality. If $f(\theta)$ has a maximum, we just replace $C_i(s)$ in Figure 1 with $-C_i(s)$. The purpose of extremum seeking is to make $\theta - \theta^*$ as small as possible, so that the output $F_o(s)[f(\theta)]$ is driven to its extremum $F_o(s)[f^*(t)]$. We pause to remark here that it is not essential that the map $f(\theta)$ be locally quadratic. The analysis below can be modified for any locally convex continuously differentiable map. We now make assumptions upon the system in Figure 1 that underlie the analysis to follow:

Assumption 2.1 *$F_i(s)$ and $F_o(s)$ are asymptotically stable and proper.*

Assumption 2.2 *$\mathcal{L}\{f^*(t)\} = \lambda_f\Gamma_f(s)$ and $\mathcal{L}\{\theta^*(t)\} = \lambda_\theta\Gamma_\theta(s)$ are strictly proper rational functions.*

This assumption obviates delta function variations in the map parameters.

Assumption 2.3 $\frac{C_o(s)}{\Gamma_f(s)}$ *and* $C_i(s)\Gamma_\theta(s)$ *are proper.*

This assumption ensures that the filters $\frac{C_o(s)}{\Gamma_f(s)}$ and $C_i(s)\Gamma_\theta(s)$ in Figure 1 can be implemented. Since $C_i(s)$ and $C_o(s)$ are at our disposal to design, we can always satisfy this assumption.

The perturbation signal $a \sin \omega t$ into the plant helps to give a measure of gradient information of the map $f(\theta)$. This is obtained by removing from the output the variation of f^* using the output filter $\frac{C_o(s)}{\Gamma_f(s)}$, and then demodulating the signal with $\sin(\omega t - \phi)$. In a sense, this can also be thought of as the online extraction of a Fourier coefficient. The analysis below makes this extraction explicit by using the modulation properties of the Laplace transform (the results used are supplied in Section 3).

2.1 Single Parameter Stability Analysis

We first provide background for the result on output extremization below. The equations describing the single parameter extremum seeking scheme in Fig. 1 are:

$$y = F_o(s)\left[f^*(t) + \frac{f''}{2}(\theta - \theta^*(t))^2\right] \tag{2.2}$$

$$\theta = F_i(s)\left[a\sin(\omega t) - C_i(s)\Gamma_\theta(s)[\xi]\right] \tag{2.3}$$

$$\xi = \sin(\omega t - \phi)\frac{C_o(s)}{\Gamma_f(s)}\left[y + n\right]. \tag{2.4}$$

For the purpose of analysis, we define the tracking error $\tilde{\theta}$ and output error $\tilde{y}$:

$$\tilde{\theta} = \theta^*(t) - \theta + \theta_0 \tag{2.5}$$

$$\theta_0 = F_i(s)\left[a\sin(\omega t)\right] \tag{2.6}$$

$$\tilde{y} = y - F_o(s)[f^*(t)]. \tag{2.7}$$

In terms of these definitions, we can restate the goal of extremum seeking as driving output error $\tilde{y}$ to a small value by tracking $\theta^*(t)$ with θ. With the present method, we cannot drive $\tilde{y}$ to zero because of the sinusoidal perturbation θ_0. We are now ready for our single parameter result:

Theorem 2.1 (Single Parameter Extremum Seeking) *For the system in Figure 1, under Assumptions 2.1, 2.2, and 2.3, the output error* $\tilde{y}$ *achieves local exponential convergence to an* $\mathrm{O}(a^2 + 1/\omega^2)$ *neighbourhood of the origin provided* $n = 0$ *and:*

1. *Perturbation frequency* ω *is sufficiently large, and* $\pm j\omega$ *is not a zero of* $F_i(s)$.

2. *Zeros of* $\Gamma_f(s)$ *that are not asymptotically stable are also zeros of* $C_o(s)$.

3. *Poles of* $\Gamma_\theta(s)$ *that are not asymptotically stable are not zeros of* $C_i(s)$.

4. $C_o(s)$ *and* $\frac{1}{1+L(s)}$ *are asymptotically stable, where*

$$L(s) = \frac{af''}{4}H_i(s)\left(\mathrm{e}^{j\phi}F_i(j\omega)H_o(s+j\omega) + \mathrm{e}^{-j\phi}F_i(-j\omega)H_o(s-j\omega)\right), \tag{2.8}$$

and $H_i(s) = C_i(s)\Gamma_\theta(s)F_i(s)$, $H_o(s) = \frac{C_o(s)}{\Gamma_f(s)}F_o(s)$.

We omit the proof as it is subsumed in the stability proof of the multiparameter extremum seeking scheme in Theorem 4.1 below. From Eqn. (2.8), we notice that $C_i(s)$ appears *linearly* in $L(s)$ (through $H_i(s) = C_i(s)\Gamma_\theta(s)F_i(s)$). This property allows systematic design using *linear control tools*. The conditions of Theorem 2.1 motivate the steps of a rigorous design algorithm below that achieves output extremization.

2.2 Single Parameter Compensator Design

In the design guidelines that follow, we set $\phi = 0$ which can be used separately for fine-tuning.

Algorithm 2.1 (Single Parameter Extremum Seeking)

1. *Select the perturbation frequency ω sufficiently large and not equal to any frequency in noise, and with $\pm j\omega$ not equal to any imaginary axis zero of $F_i(s)$.*

2. *Set perturbation amplitude a so as to obtain small steady state output error $\tilde{y}$.*

3. *Design $C_o(s)$ asymptotically stable, with zeros of $\Gamma_f(s)$ that are not asymptotically stable as its zeros, and such that $\frac{C_o(s)}{\Gamma_f(s)}$ is proper.*

4. *Design $C_i(s)$ by any linear SISO design technique such that it does not include poles of $\Gamma_\theta(s)$ that are not asymptotically stable as its zeros, $C_i(s)\Gamma_\theta(s)$ is proper, and $\frac{1}{1+L(s)}$ is asymptotically stable.*

We examine these design steps in detail:

Step 1: Since the averaging assumption is only qualitative, we may be able to choose ω only slightly larger than the plant time constants. Choice of ω equal to a frequency component persistent in the noise n can lead to a large steady state tracking error $\tilde{\theta}$. In fact, our analysis can be adapted to include a bounded noise signal satisfying $\lim_{T\to\infty} \frac{1}{T}\int_0^T n \sin\omega t dt = 0$. Finally, if $\pm j\omega$ is a zero of $F_i(s)$, we will not be able to perturb the plant.

Step 2: The perturbation amplitude a should be designed such that $a|F_i(j\omega)|$ is small; a itself may have to be large so as to produce a measurable variation in the plant output.

Step 4: We see from Algorithm 2.1 that $C_i(s)$ has to be designed such that $C_i(s)\Gamma_\theta(s)$ is proper; hence, for example, if $\Gamma_\theta(s) = \frac{1}{s^2}$, an improper $C_i(s) = 1 + d_1 s + d_2 s^2$ is permissible. In the interest of robustness, it is desirable to design $C_i(s)$ and $C_o(s)$ to ensure minimum relative degree of $C_i(s)\Gamma_\theta(s)$ and $\frac{C_o(s)}{\Gamma_f(s)}$. This will help to provide lower loop phase and thus better phase margins. Simplification of the design for $C_i(s)$ is achieved by setting $\phi = -\angle(F_i(j\omega))$, and obtaining $L(s) = \frac{af''}{4}|F_i(j\omega)|H_i(s)\left(H_o(s+j\omega) + H_o(s-j\omega)\right)$.

The attraction of extremum seeking is its ability to deal with uncertain plants. In our design, we can accommodate uncertainties in f'', $F_o(s)$, and $F_i(s)$, which appear as uncertainties in $L(s)$. Methods for treatment of these uncertainties are dealt with in texts such as [7]. Here we only show how it is possible to ensure robustness to variations in f''. Let $\widehat{f''}$ denote an a priori estimate of f''. Then we can represent $\frac{1}{1+L(s)}$ as $\frac{1}{1+L(s)} = \frac{1}{1+\left(1+\frac{\Delta f''}{\widehat{f''}}\right)P(s)}$, where $\Delta f'' = f'' - \widehat{f''}$, and $P(s) = \frac{\widehat{f''}}{f''}L(s)$, which is at our disposal because f'' in $P(s)$ gets cancelled by f'' in $L(s)$. We design $C_i(s)$ to minimize $\|\frac{P}{1+P}\|_{H_\infty}$ which maximizes the allowable $\Delta f'' < \widehat{f''}/\|\frac{P}{1+P}\|_{H_\infty}$ under which the system is still asymptotically stable.

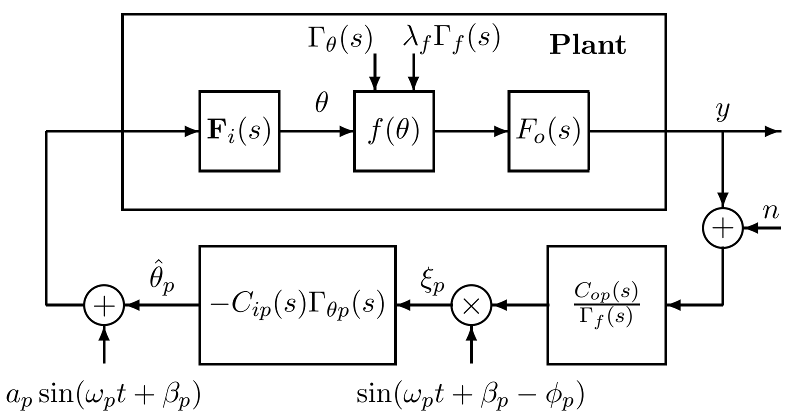

Figure 2: Multiparameter extremum seeking with $p = 1, 2, \ldots, l$. For p odd, $\omega_{p+1} = \omega_p$, $\beta_p = 0$, and $\beta_{p+1} = \pi/2$.

3 Modulation Lemmas

Lemma 3.1 ([2]) *If the transfer function $H(s)$ has all of its poles with negative real parts, then for any real ψ,*

$$H(s)\,[\sin(\omega t-\psi)] = \mathbf{Im}\left\{H(j\omega)e^{j(\omega t-\psi)}\right\} + \epsilon^{-t}, \tag{3.9}$$

where ϵ^{-t} denotes exponentially decaying terms.

Lemma 3.2 ([2]) *If the transfer functions $G(s)$ and $H(s)$ have all of their poles with negative real parts, the following statement is true for any real ϕ and any uniformly bounded $z(t)$:*

$$G(s)\,[(H(s)[\sin(\omega t-\phi)])\,z(t)]] = \mathbf{Im}\left\{e^{j(\omega t-\phi)}H(j\omega)G(s+j\omega)[z(t)]\right\} + \epsilon^{-t}.$$

Lemma 3.3 (generalization of Lemma 3.3 in [2]) *For any two rational functions $A(\cdot)$ and $B(\cdot,\cdot)$, the following is true:*

$$\begin{aligned}&\mathbf{Im}\left\{e^{j(\omega_a t-\psi)}A(j\omega_a)\right\}\mathbf{Im}\left\{e^{j(\omega_b t-\phi)}B(s,j\omega_b)[z(t)]\right\}\\ &= \frac{1}{2}\mathbf{Re}\left\{e^{j((\omega_b-\omega_a)t+\psi-\phi)}A(-j\omega_a)B(s,j\omega_b)[z(t)]\right\} - \frac{1}{2}\mathbf{Re}\left\{e^{j((\omega_b+\omega_a)t-\psi-\phi)}A(j\omega_a)B(s,j\omega_b)[z(t)]\right\}.\end{aligned}$$

4 Output Extremization in Multivariable Extremum Seeking

Figure 2 shows the multiparameter extremum seeking scheme. Analogous to the single parameter case in Section 2, we let $f(\theta)$ be a function of the form:

$$f(\theta) = f^*(t) + (\theta-\theta^*(t))^T\mathbf{P}(\theta-\theta^*(t)), \tag{4.10}$$

where $\mathbf{P}_{l\times l} = \mathbf{P}^T > 0$, $\theta = [\theta_1 \ldots \theta_l]^T$, $\theta^*(t) = [\theta_1^*(t)\ldots\theta_l^*(t)]^T$, $\mathcal{L}\{\theta^*(t)\} = \Gamma_\theta(s) = [\lambda_1\Gamma_{\theta 1}(s)\ldots \lambda_l\Gamma_{\theta l}(s)]^T$, and $\mathcal{L}\{f^*(t)\} = \lambda_f\Gamma_f(s)$. Any vector function $f(\theta)$ with a quadratic minimum at θ^* can be approximated by Eqn. (4.10). In seeking maxima, i.e., $\mathbf{P} < 0$, we only need to replace $C_{ip}(s)$ with $-C_{ip}(s)$. Further, the method need not confine itself to seeking only extrema; convergence to saddle points, and any points with zero first derivative may be attained using the designs developed here simply by setting $C_{ip}(s)$ the same sign as P_{pp}. We further note here that we propose diagonal compensation in the scheme in Figure 2 for two reasons: while we can obtain a MISO sensitivity design problem, there are no systematic means of multiparameter design when there is an unknown matrix gain ($\mathbf{P}$) in the plant; use of a MIMO compensator also leads to an $\mathrm{O}(l)$ increase in the steady-state output deviation from the extremum.

The broad principle of using m frequencies for identification/tracking of $2m$ parameters holds here. Forcing frequencies $\omega_1 < \omega_3 < \ldots < \omega_{2m-1}$ are used, where $m = \left[\frac{l+2}{2}\right]$ ($[x]$ is the greatest integer less than x). We make assumptions analogous to the single parameter case:

Assumption 4.1 *$\mathbf{F}_i(s) = [F_{i1}(s) \ldots F_{il}(s)]^T$ and $F_o(s)$ are asymptotically stable and proper.*

Assumption 4.2 *$\Gamma_\theta(s)$ and $\Gamma_f(s)$ are strictly proper.*

Assumption 4.3 *$C_{ip}(s)\Gamma_{\theta p}(s)$ and $\frac{C_{op}(s)}{\Gamma_f(s)}$ are proper for all $p = 1, 2, \ldots, l$.*

In the multiparameter case, we make an additional assumption upon the perturbation frequencies:

Assumption 4.4 ([4]) *$\omega_p + \omega_q \neq \omega_r$ for any $p, q, r = 1, 2, \ldots, l$.*

The purpose of this assumption is to preclude bias terms arising from demodulation in the case of a quadratic nonlinearity. We can always satisfy this assumption since the choice of frequencies is at our disposal. We expatiate further upon this assumption at the end of this section.

The equations governing the p^{th} loop of the multiparameter scheme in Figure 2 are as follows:

$$\begin{aligned} y &= F_o(s)\left[f^* + (\theta - \theta^*)^T \mathbf{P}(\theta - \theta^*)\right] && (4.11)\\ \theta_p &= F_{ip}(s)\left[a_p \sin(\omega_p t + \beta_p) - C_{ip}(s)\Gamma_{\theta p}(s)[\xi_p]\right] && (4.12)\\ \xi_p &= \sin(\omega_p t + \beta_p - \phi_p)\frac{C_{op}(s)}{\Gamma_f(s)}[y + n], && (4.13) \end{aligned}$$

where

$$\beta_p = \begin{cases} 0 & , \quad p \text{ odd} \\ \frac{\pi}{2} & , \quad p \text{ even} \end{cases} \tag{4.14}$$

and, for p odd, $\omega_{p+1} = \omega_p$. The definitions of tracking error $\tilde{\theta}_p$ and output error $\tilde{y}$ analogous to the single parameter case are:

$$\begin{aligned} \tilde{\theta}_p &= \theta_p^* - \theta_p + \theta_{0p}; \quad \tilde{\theta} = [\tilde{\theta}_1 \ldots \tilde{\theta}_l]^T && (4.15)\\ \theta_{0p} &= F_{ip}(s)[a_p \sin(\omega_p t + \beta_p)]; \quad {\theta_0}^T = [\theta_{01} \ldots \theta_{0l}]^T && (4.16)\\ \tilde{y} &= y - F_o(s)[f^*] = F_o(s)\left[(\theta - \theta^*)^T \mathbf{P}(\theta - \theta^*)\right]. && (4.17) \end{aligned}$$

We now state our result on multiparameter output extremization:

Theorem 4.1 (Multiparameter Extremum Seeking) *For the system in Figure 2, under Assumptions 4.1–4.4, the output error $\tilde{y}$ achieves local exponential convergence to an $\mathrm{O}(1/\omega_1^2 + l\sum_{p=1}^{l} a_p^2)$ neighbourhood of zero provided $n = 0$ and:*

1. *Perturbation frequencies $\omega_1 < \omega_3 < \ldots < \omega_{2m-1}$ are rational, sufficiently large, and $\pm j\omega_p$ is not a zero of $F_{ip}(s)$.*

2. *Zeros of $\Gamma_f(s)$ that are not asymptotically stable are also zeros of $C_{op}(s)$, for all $p = 1, \ldots, l$.*

3. *Poles of $\Gamma_{\theta p}(s)$ that are not asymptotically stable are not zeros of $C_{ip}(s)$, for any $p = 1, \ldots, l$.*

4. *$C_{op}(s)$ are asymptotically stable for all $p = 1, \ldots, l$ and $\frac{1}{\det(\mathbf{I}_l + \mathbf{X}(s))}$ is asymptotically stable, where $X_{pq}(s)$ denote the elements of $\mathbf{X}(s)$ and*

$$X_{pq}(s) = P_{pq} a_p L_p(s) + P_{p+\delta,q} a_{p+\delta} M_p(s), \quad q = 1, \ldots, l \tag{4.18}$$

$$L_p(s) = \frac{1}{2} H_{ip}(s) \left[\mathrm{e}^{j\phi_p} F_{ip}(j\omega_p) H_{op}(s + j\omega_p) + \mathrm{e}^{-j\phi_p} F_{ip}(-j\omega_p) H_{op}(s - j\omega_p) \right] \tag{4.19}$$

$$\begin{aligned} M_p(s) = & \frac{1}{2} H_{ip}(s) \Big[\mathrm{e}^{j(\phi_p + \delta\frac{\pi}{2})} F_{i,p+\delta}(j\omega_p) H_{op}(s + j\omega_p) \\ & + \mathrm{e}^{-j(\phi_p + \delta\frac{\pi}{2})} F_{i,p+\delta}(-j\omega_p) H_{op}(s - j\omega_p) \Big], \end{aligned} \tag{4.20}$$

where $\delta = 1$ for p odd and $\delta = -1$ for p even, and $H_{ip}(s) = C_{ip}(s)\Gamma_{\theta p}(s) F_{ip}(s)$ and $H_{op}(s) = \frac{C_{op}(s)}{\Gamma_f(s)} F_o(s)$.

Proof: We expand $\tilde{\theta}_n$ in Eqn. (4.15), substituting for θ_n, ξ_n, and y from Eqns. (4.12), (4.13), and (4.11) respectively and get:

$$\begin{aligned} \tilde{\theta}_n &= \theta_n^* + H_{in}(s) \left[\sin(\omega_n t + \beta_n - \phi_n) H_{on}(s)[f^* + (\theta - \theta^*)^T \mathbf{P} (\theta - \theta^*)] \right] \\ &= \theta_n^* + H_{in}(s) \left[\sin(\omega_n t + \beta_n - \phi_n) H_{on}(s)[f^* + (\tilde{\theta} - \theta_0)^T \mathbf{P} (\tilde{\theta} - \theta_0)] \right], \end{aligned} \tag{4.21}$$

using $\theta - \theta^* = \theta_0 - \tilde{\theta}$ from Eqn. (4.15). Here, in addition to terms encountered in the single parameter case, we have to consider linear terms and higher order terms that arise due to coupling from the quadratic form:

$$(\tilde{\theta} - \theta_0)^T \mathbf{P} (\tilde{\theta} - \theta_0) = \sum_{p=1}^{l} \sum_{q=1}^{l} P_{pq} \left(\tilde{\theta}_p \tilde{\theta}_q + \theta_{0p}\theta_{0q} - \tilde{\theta}_p \theta_{0q} - \tilde{\theta}_q \theta_{0p} \right) \tag{4.22}$$

The term containing $f^*(t)$, and $\theta_{0p}\theta_{0q}$ in Eqn. (4.21) can be simplified using Lemma 3.1 as in the proof of Theorem 2.1 in the Appendix, using Assns. 4.1, 4.2, 4.3, and asymptotic stability of $C_{on}(s)$:

$$\sin(\omega_n t + \beta_n - \phi_n) H_{on}(s)[f^* + \sum_{p=1}^{l} \sum_{q=1}^{l} P_{pq} \theta_{0p} \theta_{0q}] = w_n(t) + \epsilon^{-t}, \tag{4.23}$$

where ϵ^{-t} denotes exponentially decaying terms, and

$$\begin{aligned} w_n(t) = & \sum_{p=1}^{l} \sum_{q=1}^{l} P_{pq} a_p a_q \{ C_{pq1} \sin[(\omega_n + \omega_p + \omega_q)t - \mu_{pq1}] + C_{pq2} \sin[(\omega_n + \omega_p - \omega_q)t - \mu_{pq2}] \\ & + C_{pq3} \sin[(\omega_n - \omega_p + \omega_q)t - \mu_{pq3}] + C_{pq4} \sin[\omega_n - \omega_p - \omega_q)t - \mu_{pq4}] \}, \end{aligned} \tag{4.24}$$

where

$$C_{pq1} = C_{pq4} = |F_{ip}(j\omega_p) F_{iq}(j\omega_q) H_{on}(j(\omega_p + \omega_q))| \tag{4.25}$$

$$C_{pq2} = C_{pq3} = |F_{ip}(j\omega_p) F_{iq}(j\omega_q) H_{on}(j(\omega_p - \omega_q))|, \tag{4.26}$$

and the constants μ_{pqr}, $r = 1,\ldots,4$ depend upon ϕ_n and the phases of $F_{ip}(j\omega_p)$, $F_{iq}(j\omega_q)$, and $H_{on}(j(\omega_p \pm \omega_q))$. The function $w_n(t)$ is of order $\mathrm{O}(\sum_{p=1}^{l}\sum_{q=1}^{l} a_p a_q) = \mathrm{O}(l\sum_{p=1}^{l} a_p^2)$ and does not contain constant terms since from Assn. 4.4, $\omega_p + \omega_q \neq \omega_r$ for any $p,q,r = 1,\ldots,l$. Using Eqns. (4.22), (4.23), and symmetry of $\mathbf{P}$, we can now rewrite Eqn. (4.21) as follows:

$$\tilde{\theta}_n = \theta_n^* + H_{in}(s)\left[\sin(\omega_n t + \beta_n - \phi_n)H_{on}(s)\left[\sum_{p=1}^{l}\sum_{q=1}^{l} 2P_{pq}\tilde{\theta}_q\theta_{0p} + P_{pq}\tilde{\theta}_p\tilde{\theta}_q\right] + w_n(t) + \epsilon^{-t}\right]. \quad (4.27)$$

Using asymptotic stability of $H_{on}(s)$ (from Assns. 4.1, 4.2, 4.3, and asymptotic stability of $C_{on}(s)$) and applying Lemmas 3.1, 3.2, and 3.3 in succession to the term containing $\tilde{\theta}_q\theta_{0p}$ in Eqn. (4.27), we get[2]:

$$H_{in}(s)\left[\sin(\omega_n t + \beta_n - \phi_n)H_{on}(s)[2P_{pq}\tilde{\theta}_q\theta_{0p}]\right] = H_{in}(s)\left[\mathcal{T}_{npq}[\tilde{\theta}_q] - \mathcal{K}_{npq}[\tilde{\theta}_q]\right], \quad (4.28)$$

where

$$\mathcal{T}_{npq}[\tilde{\theta}_q] = P_{pq}a_p\left[\mathrm{Re}\left\{\mathrm{e}^{j((\omega_n+\omega_p)t+\beta_n-\phi_n+\beta_p)}v_{pq}\right\}\right] \quad (4.29)$$
$$\mathcal{K}_{npq}[\tilde{\theta}_q] = P_{pq}a_p\left[\mathrm{Re}\left\{\mathrm{e}^{j((\omega_p-\omega_n)t-\beta_n+\phi_n+\beta_p)}v_{pq}\right\}\right] \quad (4.30)$$
$$v_{pq} = F_{ip}(j\omega_p)H_{on}(s+j\omega_p)[\tilde{\theta}_q]. \quad (4.31)$$

Since we are proving a local result, we drop the second order terms containing $\tilde{\theta}_p\tilde{\theta}_q$ and rewrite Eqn. (4.27) as follows after moving terms linear in $\tilde{\theta}_q$ to the left hand side:

$$\tilde{\theta}_n + H_{in}(s)\left[\sum_p\sum_q(\mathcal{K}_{npq} - \mathcal{T}_{npq})[\tilde{\theta}_q]\right] = \theta_n^* + H_{in}(s)\left[w_n(t) + \epsilon^{-t}\right]. \quad (4.32)$$

We consider below the low frequency second terms $H_{in}(s)[\mathcal{K}_{npq}]$ in Eqn. (4.28) in the following cases to explicitly show the time invariant terms:

1. $q = n$, $p = n$: We get the term, $H_{in}(s)[\mathcal{K}_{npq}] = P_{nn}a_nL_n(s)$.
2. $q = n$, $p = n+\delta$: We get $H_{in}(s)[\mathcal{K}_{npq}] = P_{n+\delta,n}a_{n+\delta}M_n(s)$.
3. $q \neq n$, $p = n$: We get a term $H_{in}(s)[\mathcal{K}_{npq}] = P_{nq}a_nL_n(s)$.
4. $q \neq n$, $p = n+\delta$: We get a term $H_{in}(s)[\mathcal{K}_{npq}] = P_{n+\delta,q}a_{n+\delta}M_n(s)$.
5. $q \neq n, p \neq n+\delta$: $\mathcal{K}_{npq}$ is time-varying.

Using the above, we can rewrite Eqn. (4.27) in a form that shows separately the time invariant terms:

$$\tilde{\theta}_n + \sum_q X_{nq}(s)[\tilde{\theta}_q] + H_{in}(s)\left[\sum_{p\neq n,n+\delta}\sum_q(\mathcal{K}_{npq} - \mathcal{T}_{npq})[\tilde{\theta}_q] - \sum_{p=n,n+\delta}\sum_q \mathcal{T}_{npq}[\tilde{\theta}_q]\right]$$
$$= \theta_n^* + H_{in}(s)\left[w_n(t) + \epsilon^{-t}\right]. \quad (4.33)$$

[2]Note that Eqn. (4.28) contains additional terms of the form $H_{in}(s)[\sin(\omega_n t + \beta_n - \phi_n)H_{on}(s)[\epsilon^{-t}\tilde{\theta}_q]]$ which comes from ϵ^{-t} in $\theta_{0p}(t) = a\mathbf{Im}\{F_{ip}(j\omega_p)\mathrm{e}^{j\omega_p t}\} + \epsilon^{-t}$. We drop this term from subsequent analysis because it does not affect closed loop stability or asymptotic performance. It can be accounted for in three ways. One is to perform averaging over an infinite time interval in which all exponentially decaying terms disappear. The second way is to treat $\epsilon^{-t}\tilde{\theta}_q$ as a vanishing perturbation via Corollary 5.4 in Khalil [1], observing that ϵ^{-t} is integrable. The third way is to express ϵ^{-t} in state space format and let $\epsilon^{-t}\tilde{\theta}_q$ be dominated by other terms in a local Lyapunov analysis.

The rest of the proof shows how only the time invariant terms in Eqn. (4.33) need be considered for stability of the system. We now proceed to put the Eqns. (4.32) in a form suitable for applying averaging. Dividing both sides of Eqn. (4.32) with $\det(\mathbf{I}+\mathbf{X}(s))$, we get

$$\frac{1}{\det(\mathbf{I}+\mathbf{X}(s))}[\tilde{\theta}_n] + Y_{in}(s)\left[\sum_p\sum_q(\mathcal{K}_{npq}-\mathcal{T}_{npq})\,[\tilde{\theta}_q]\right] = \frac{1}{\det(\mathbf{I}+\mathbf{X}(s))}[\theta_n^*] + Y_{in}(s)\left[w_n(t)+\epsilon^{-t}\right], \tag{4.34}$$

where $Y_{in}(s) = \frac{H_{in}(s)}{\det(\mathbf{I}+\mathbf{X}(s))} = \frac{\text{num}\{Y_{in}(s)\}}{\text{num}\{\det(\mathbf{I}+\mathbf{X}(s))\}}$ is asymptotically stable because poles of $H_{in}(s)$ that are not asymptotically stable are cancelled by zeros in $\frac{1}{\det(\mathbf{I}+\mathbf{X}(s))}$ (using condition 3 of Theorem 4.1), and $\frac{1}{\det(\mathbf{I}+\mathbf{X}(s))}$ is asymptotically stable. By noting also that zeros in $\frac{1}{\det(\mathbf{I}+\mathbf{X}(s))}$ cancel poles in $\theta_n^*(s) = \lambda_\theta\Gamma_{\theta n}(s)$ that are not asymptotically stable (using condition 3 of Theorem 4.1), and using asymptotic stability of $\frac{1}{\det(\mathbf{I}+\mathbf{X}(s))}$, we get

$$\frac{1}{\det(\mathbf{I}+\mathbf{X}(s))}[\tilde{\theta}_n] + Y_{in}(s)\left[\sum_p\sum_q(\mathcal{K}_{npq}-\mathcal{T}_{npq})\,[\tilde{\theta}_q]\right] = \varepsilon + Y_{in}(s)\,[w_n(t)] \tag{4.35}$$

$$\varepsilon = \left[\frac{1}{\det(\mathbf{I}+\mathbf{X}(s))}[\theta_n^*] + Y_{in}(s)\left[\epsilon^{-t}\right]\right], \tag{4.36}$$

where ε is exponentially decaying. Multiplying both sides of Eqn (4.35) with $\text{num}\{\det(\mathbf{I}+\mathbf{X}(s))\}$, and expanding operators $\mathcal{K}_{npq}$ and $\mathcal{T}_{npq}$, we rewrite Eqn. (4.35) as

$$\begin{aligned}&\text{den}\{\det(\mathbf{I}+\mathbf{X}(s))\}[\tilde{\theta}_n]\\ &+\text{num}\{Y_{in}(s)\}\left[\sum_p\sum_q P_{pq}a_p\left(\text{Re}\left\{\text{e}^{n((\omega_p-\omega_n)t-\beta_n+\phi_n+\beta_p)}v_{pq}\right\} - \text{Re}\left\{\text{e}^{j((\omega_n+\omega_p)t+\beta_n-\phi_n+\beta_p)}v_{pq}\right\}\right)\right]\\ &= \text{num}\{\det(\mathbf{I}+\mathbf{X}(s))\}[\varepsilon] + \text{num}\{Y_{in}(s)\}\,[w_n(t)]\,.\end{aligned} \tag{4.37}$$

Now, there exist polynomials $Z_{j(\omega_p-\omega_n)}(s)$, $Z_{j(\omega_p+\omega_n)}(s)$ whose order is the same as that of $\text{num}\{Y_{in}(s)\}$, whose coefficients are dependent on $(\omega_p-\omega_n)$, $(\omega_p+\omega_n)$ respectively, and such that

$$\begin{aligned}&\text{num}\{Y_{in}(s)\}\left[\sum_p\sum_q P_{pq}a_p\left(\text{e}^{j((\omega_p-\omega_n)t-\beta_n+\phi_n+\beta_p)}v_{pq} - \text{e}^{j((\omega_n+\omega_p)t+\beta_n-\phi_n+\beta_p)}v_{pq}\right)\right]\\ &=\sum_p\sum_q P_{pq}a_p\left(\text{e}^{j((\omega_p-\omega_n)t-\beta_n+\phi_n+\beta_p)}Z_{j(\omega_p-\omega_n)}(s)[v_{pq}]\right.\\ &\quad\left.-\text{e}^{j((\omega_n+\omega_p)t+\beta_n-\phi_n+\beta_p)}Z_{j(\omega_p+\omega_n)}(s)[v_{pq}]\right),\end{aligned} \tag{4.38}$$

and $Z_0(s) = \text{num}\{Y_{in}(s)\}$. Hence we can write the system of Eqns. (4.37) as

$$\begin{aligned}&\text{den}\{\det(\mathbf{I}+\mathbf{X}(s))\}[\tilde{\theta}_n] + \sum_p\sum_q P_{pq}a_p\left(\text{Re}\left\{\text{e}^{j((\omega_p-\omega_n)t+\phi_n+\beta_p)}Z_{j(\omega_p-\omega_n)}(s)[v_{pq}]\right\}\right.\\ &\quad\left.-\text{Re}\left\{\text{e}^{j((\omega_n+\omega_p)t-\phi_n+\beta_p)}Z_{j(\omega_p+\omega_n)}(s)[v_{pq}]\right\}\right)\\ &=\text{num}\{\det(\mathbf{I}+\mathbf{X}(s))\}[\varepsilon] + \text{num}\{Y_{in}(s)\}\,[w_n(t)]\,.\end{aligned} \tag{4.39}$$

The system of Eqns. (4.39) can be written as a set of time-varying linear differential equations, and this, along with Eqns. (4.31) for v_{pq} and $\bar{v}_{pq}$ (for all $n,p,q=1,\dots,l$) can be put into the state space form:

$$\dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x} + \mathbf{A}_{12}\mathbf{x}_e + \mathbf{B}\mathbf{w}(t);\quad \tilde{\theta} = \mathbf{C}\mathbf{x} + \mathbf{C}_{12}\mathbf{x}_e + \mathbf{D}\mathbf{w}(t) \tag{4.40}$$

$$\dot{\mathbf{x}}_e = \mathbf{A}_e\mathbf{x}_e, \tag{4.41}$$

where $\mathbf{w}(t)^T = [w_1(t), \ldots, w_l(t)]$, and Eqn. (4.41) is a representation for the exponentially decaying term ε. As all forcing frequencies $\omega_1, \ldots, \omega_m$, and consequently their linear combinations, are rational, there exists a period T, which is the lowest common multiple of all the time-periods in the system, such that the system in Eqn. (4.40) is T-periodic[3]. We get Eqns. (4.40), (4.41) into the standard form for averaging by using the transformation $\tau = \omega_1 t$, and then averaging the right hand side of the equations w.r.t time from 0 to T, i.e., $\frac{1}{T}\int_0^T (\cdot) d\tau$ treating states $\mathbf{x}$, $\mathbf{x}_e$ as constant. The averaged equations are:

$$\frac{d\mathbf{x}_{av}}{d\tau} = \frac{1}{\omega_1}\left(\mathbf{A}_{av}\mathbf{x}_{av} + \mathbf{A}_{12}\mathbf{x}_{eav}\right), \ \tilde{\theta}_{av} = \mathbf{C}\mathbf{x}_{av} + \mathbf{C}_{12}\mathbf{x}_{eav} \tag{4.42}$$

$$\frac{d\mathbf{x}_{eav}}{d\tau} = \frac{1}{\omega_1}\mathbf{A}_e\mathbf{x}_{eav}, \tag{4.43}$$

where $\mathbf{A}_{av} = \frac{1}{T}\int_0^T \mathbf{A}(\tau)d\tau$. This yields

$$\begin{aligned}
&\text{den}\{\det(\mathbf{I}+\mathbf{X}(s))\}[\tilde{\theta}_{nav}] + \text{num}\{Y_{in}(s)\}\left[\sum_{p=n,n+\delta}\sum_q P_{pq}a_p \text{Re}\left\{\mathrm{e}^{j(\phi_n-\beta_n+\beta_p)}[v_{pqav}]\right\}\right] \\
&= \text{den}\{\det(\mathbf{I}+\mathbf{X}(s))\}\left[\tilde{\theta}_{nav} + \sum_q X_{nq}(s)[\tilde{\theta}_{qav}]\right] \\
&= \text{num}\{\det(\mathbf{I}+\mathbf{X}(s))\}[\epsilon^{-t}],
\end{aligned} \tag{4.44}$$

in the original time-scale, using $Z_0(s) = \text{num}\{Y_{in}(s)\}$ and substituting for v_{pq} from Eqn. (4.31). Expanding the right hand side of Eqn. (4.44) as

$$\text{num}\{\det(\mathbf{I}+\mathbf{X}(s))\}[\epsilon^{-t}] = \text{num}\{\det(\mathbf{I}+\mathbf{X}(s))\}\left[\frac{1}{\det(\mathbf{I}+\mathbf{X}(s))}[\theta_n^*] + Y_{in}(s)\left[\epsilon^{-t}\right]\right],$$

and dividing both sides with $\text{num}\{\det(\mathbf{I}+\mathbf{X}(s))\}$ (which is asymptotically stable), we get

$$\begin{aligned}
\frac{1}{\det(\mathbf{I}+\mathbf{X}(s))}\left[\tilde{\theta}_{nav} + \sum_q X_{nq}(s)[\tilde{\theta}_{qav}]\right] &= \frac{1}{\det(\mathbf{I}+\mathbf{X}(s))}[\theta_n^*] + Y_{in}(s)\left[\epsilon^{-t}\right] \\
&= \frac{1}{\det(\mathbf{I}+\mathbf{X}(s))}\left[\theta_n^* + H_{in}(s)\left[\epsilon^{-t}\right]\right],
\end{aligned} \tag{4.45}$$

using $Y_{in}(s) = \frac{H_{in}(s)}{\det(\mathbf{I}+\mathbf{X}(s))}$. Eqn. (4.45) represents a system of equations that can be written in matrix form as

$$\tilde{\theta}_{av} = (\mathbf{I}+\mathbf{X}(s))^{-1}\left[\theta^* + \mathbf{H}_i(s)\left[\epsilon^{-t}\right]\right], \tag{4.46}$$

where $\mathbf{H}_i(s) = [H_{i1}(s), \ldots, H_{il}(s)]$. $\tilde{\theta}_{av}$ decays to zero because $(\mathbf{I}+\mathbf{X}(s))^{-1}$ is asymptotically stable (owing to asymptotic stability of $\frac{1}{\det(\mathbf{I}+\mathbf{X}(s))}$), and zeros in $(\mathbf{I}+\mathbf{X}(s))^{-1}$ cancel unstable poles in θ^* and $\mathbf{H}_i(s)$. Hence, by a standard averaging theorem such as Theorem 8.3 in Khalil [1], we see that if ω_p, a_p, ϕ_p, $C_{ip}(s)$ and $C_{op}(s)$ for all $p = 1, \ldots, l$ are such that $\frac{1}{\det(\mathbf{I}+\mathbf{X}(s))}$ is asymptotically stable, $C_{op}(s)$ is asymptotically stable, ω_1 is sufficiently large relative to parameters of the state-space

[3]The frequencies $\omega_1, \ldots, \omega_m$, being rational, can be written as $\omega_1, \frac{p_1}{q_1}\omega_1, \frac{p_2}{q_2}\omega_1, \ldots, \frac{p_{m-1}}{q_{m-1}}\omega_1$. The time periods in the system are $\frac{2\pi}{\omega_p}$, $\frac{2\pi}{(\omega_p-\omega_q)}, \omega_p \neq \omega_q$, $\frac{2\pi}{(\omega_p+\omega_q)}$, $\frac{2\pi}{(\omega_p+\omega_q-\omega_r)}$, and $\frac{2\pi}{(\omega_p+\omega_q+\omega_r)}$, $p, q, r = 1, \ldots, m$, all of which are rational multiples of 2π. Thus, T can be calculated as the lowest common multiple of these time-periods.

representation, solutions starting from small initial conditions converge exponentially to a periodic solution in an $O(1/\omega_1)$ neighbourhood of zero. Hence, $\tilde{\theta}$ goes to $O(1/\omega_1)$. Further, the output error $\tilde{y}$ decays to $O(1/\omega_1^2 + l\sum_{p=1}^{l} a_p^2)$:

$$\begin{aligned}\tilde{y} &= F_o(s)[(\theta-\theta^*)^T\mathbf{P}(\theta-\theta^*)] = F_o(s)[(\tilde{\theta}-\theta_0)^T\mathbf{P}(\tilde{\theta}-\theta_0)] \\ &= F_o(s)[\tilde{\theta}^T\mathbf{P}\tilde{\theta} + \theta_0^T\mathbf{P}\theta_0 - 2\tilde{\theta}^T\mathbf{P}\theta_0] \\ &= O(1/\omega_1^2 + \sum_{p=1}^{l}\sum_{q=1}^{l} a_p a_q) = O(1/\omega_1^2 + l\sum_{p=1}^{l} a_p^2),\end{aligned} \tag{4.47}$$

Q. E. D.

We have proved Theorem 4.1 for the case where a single frequency is used for tracking of two parameters. Because of the coupling this introduces through the $M_p(s)$ terms in each of the $X_{pq}(s)$, the process of multiparameter design may become difficult. When it is possible to use more frequencies, the design may be simpler. Hence, we provide here a corollary to Theorem 4.1 when a forcing frequency is dedicated to tracking only one parameter instead of two:

Corollary 4.1 *If forcing frequencies $\omega_1 < \omega_2 < \ldots < \omega_s$, $2m-1 < s \leq l$ are chosen for the scheme in Figure 2, and all the other conditions of Theorem 4.1 hold, its result also holds with $X_{pq}(s) = P_{pq}a_pL_p(s)$ for each p where $\omega_p \neq \omega_r$ for any $r \neq p$, and $X_{pq}(s)$ is given by Eqn (4.18) otherwise.*

The result follows from the fact that the coupling terms $M_p(s)$ vanish when a forcing frequency is used only for one tracking loop.

Now, we briefly discuss Assumption 4.4. From Eqns. (4.23), (4.24) it is clear that the assumption precludes constant terms in $w_n(t)$ only when the nonlinearity is quadratic. For a general nonlinearity, the frequencies can be designed incommensurate, and the analysis result arrived at by infinite time averaging. Even without this assumption, exponential convergence of $\tilde{y}$ to a neighbourhood of the origin can be attained if the constant terms in $w_n(t)$ are small, but the analysis is longer, since we would have to linearize equations for $\tilde{\theta}_n$ about those constant terms, and then perform averaging.

5 Multiparameter Design

The process of design for the multiparameter case can be divided into the following sequential steps: selection of frequencies $\omega_1, \omega_2, \ldots, \omega_l$, selection of perturbation amplitudes $a_1, a_2, \ldots, a_l$, design of compensators $C_{op}(s)$ and $C_{ip}(s)$ for each p, to satisfy the conditions of Theorem 4.1.

The complexity of multiparameter design arises from the need for asymptotic stabilization of $\frac{1}{\det(\mathbf{I}+\mathbf{X}(s))}$, which is intricately coupled. Methods of decentralized control, as those in [5] do not apply to our problem because the coupling between different subsystems enters through the compensators $C_{ip}(s)$ due to a single output being used for measurement. We propose here a method of reducing the general problem to allow independent SISO design of each of the compensators $C_{ip}(s)$. The method involves domination of the off diagonal terms in $\mathbf{I}+\mathbf{X}(s)$ by the diagonal terms, and may be termed *diagonal domination design*.

l	2	3	4	5	6	7	8	9	10
ρ_l^*	1	0.5	0.3239	0.2367	0.1855	0.1520	0.1284	0.1111	0.0978

Table 1: Design difficulty in general design increases with dimension l

Proposition 5.1 *Let ρ_l^* denote the unique solution in the interval $(0,1]$ of the polynomial equation*[4] $\mathrm{per}\,(\Sigma(\rho)) = 2$, $\Sigma(\rho) = \begin{pmatrix} 1 & \rho & \cdots & \rho \\ \rho & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \rho \\ \rho & \cdots & \rho & 1 \end{pmatrix}_{l\times l}$. *If $\frac{X_{pp}(s)}{1+X_{pp}(s)}$ are asymptotically stable and $\|\frac{X_{pq}}{1+X_{pp}}\|_{H_\infty} < \rho_l^*$ for all $p \neq q$, then $\frac{1}{\det(\mathbf{I}+\mathbf{X}(s))}$ is asymptotically stable, where $X_{pq}(s)$ are defined in Eqn. (4.18).*

From the definition of the permanent of a matrix, $\mathrm{per}\,(\Sigma(\rho))$ is a polynomial with positive integer coefficients and thus a monotonically increasing function of ρ when $\rho > 0$. Since $\mathrm{per}\,(\Sigma(0)) = 1$ and $\mathrm{per}\,(\Sigma(1)) \geq 2$, we have by continuity, a unique solution to the equation $\mathrm{per}\,(\Sigma(\rho)) = 2$ in the interval $(0,1]$. The equation $\mathrm{per}\,(\Sigma(\rho)) = 2$ expands as $\rho^2 = 1$ and $2\rho^3 + 3\rho^2 = 1$ in two and three dimensions, respectively, yielding $\rho_2^* = 1$, and $\rho_3^* = 0.5$. Thus the crux of Proposition 5.1 is that if the transfer functions $\frac{X_{pq}(s)}{1+X_{pp}(s)}$ are norm bounded by a number ρ_l^* that *depends only upon the dimension of the problem* l, we have asymptotic stability of $\frac{1}{\det(\mathbf{I}+\mathbf{X}(s))}$. For convenience, we list values of ρ_l^* upto $l = 10$ in Table 1. It can be shown that $\frac{1}{\rho_l^*} \leq \sqrt{l! - 1}$.

Proof of Proposition 5.1: We first rewrite the determinant of $(\mathbf{I} + \mathbf{X}(s))$ as follows:

$$\det(\mathbf{I}+\mathbf{X}(s)) = \det \begin{pmatrix} 1 & \frac{X_{12}(s)}{1+X_{11}(s)} & \frac{X_{13}(s)}{1+X_{11}(s)} & \cdots & \frac{X_{1l}(s)}{1+X_{11}(s)} \\ \frac{X_{21}(s)}{1+X_{22}(s)} & 1 & \frac{X_{23}(s)}{1+X_{22}(s)} & \cdots & \frac{X_{2l}(s)}{1+X_{22}(s)} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \frac{X_{l1}(s)}{1+X_{ll}(s)} & \frac{X_{l2}(s)}{1+X_{ll}(s)} & \frac{X_{l3}(s)}{1+X_{ll}(s)} & \cdots & 1 \end{pmatrix} \prod_{p=1}^{l}(1+X_{pp}(s)) \quad (5.48)$$

$$= \det(\mathbf{Y}(s))\prod_{p=1}^{l}(1+X_{pp}(s)) \quad (5.49)$$

$$= (1+W(s))\prod_{p=1}^{l}(1+X_{pp}(s)), \quad (5.50)$$

where Eqns. (5.48), (5.49), and (5.50) define $\mathbf{Y}(s)$ and $W(s)$. Therefore, we have

$$\frac{1}{\det(\mathbf{I}+\mathbf{X}(s))} = \frac{1}{(1+W(s))\prod_{p=1}^{l}(1+X_{pp}(s))}. \quad (5.51)$$

Now, we note that as $\frac{X_{pp}(s)}{1+X_{pp}(s)} \in H_\infty$ for each p, each of $\frac{1}{1+X_{pp}(s)}$ is asymptotically stable. Hence, if we can achieve $\|W\|_{H_\infty} < 1$, we have asymptotic stability of $\frac{1}{\det(\mathbf{I}+\mathbf{X}(s))}$. Using $1 + W(s) =$

[4]The permanent of a matrix $\mathbf{A}$ is defined as $\mathrm{per}\mathbf{A} = \sum_\sigma \prod_{i=1}^n a_{i,\sigma(i)}$, where the sum runs over all $n!$ permutations σ of $\{1,\ldots,n\}$, and $\sigma(i)$ is the i^{th} element of the permutation σ. We note that the permanent of a matrix is simply the sum of all the terms in its determinant, with all the products $\prod_{i=1}^n a_{i,\sigma(i)}$ entering with coefficient 1 instead of a power of -1.

$\det(\mathbf{Y}(s))$, we have:

$$W(s) = \sum_{\sigma} \mathrm{sgn}\sigma \prod_{i=1}^{l} y_{i,\sigma(i)}(s), \tag{5.52}$$

where the sum runs over $l! - 1$ permutations σ of $\{1, \ldots, l\}$ excluding the permutation $\{1, \ldots, l\}$ to account for the cancellation of unity in Eqn. (5.52), sgnσ is positive or negative depending upon whether the number of pairwise interchanges needed to arrive at the permutation σ from the permutation $\{1, \ldots, l\}$ is even or odd, and $\sigma(i)$ is the i^{th} element of the permutation σ.

We are now in a position to bound the H_∞ norm of $W(s)$ through repeated application of the triangle inequality and the submultiplicative property of the H_∞ norm:

$$\|W\|_{H_\infty} \leq \sum_{\sigma} \| \prod_{i=1}^{l} y_{i,\sigma(i)}(s) \|_{H_\infty} \leq \sum_{\sigma} \prod_{i=1}^{l} \|y_{i,\sigma(i)}(s)\|_{H_\infty}. \tag{5.53}$$

Substituting $\|\frac{X_{pq}}{1+X_{pp}}\|_{H_\infty} < \rho_l^*$ for all p, q, in Eqn. (5.53), and using the fact that ρ_l^* is the unique solution of the equation $\mathrm{per}\,(\Sigma(\rho)) = 2$ in the interval $(0, 1]$, we have

$$\|W\|_{H_\infty} < \mathrm{per}\Sigma(\rho_l^*) - 1 = 1. \tag{5.54}$$

From asymptotic stability of each of $\frac{1}{1+X_{pp}(s)}$, and of $\frac{1}{1+W(s)}$, we have asymptotic stability of $\frac{1}{\det(\mathbf{I}+\mathbf{X}(s))}$ from Eqn. (5.51). Q.E.D.

While Proposition 5.1 provides a sufficient condition for asymptotic stability of $\frac{1}{\det(\mathbf{I}+\mathbf{X}(s))}$, it does not provide means to guarantee it. Hence the problem has now to be transformed to permit systematic design of the compensators $C_{ip}(s)$ to achieve $\|\frac{X_{pq}}{1+X_{pp}}\|_{H_\infty} < \rho_l^*$ for all p, q. To this end, we express the off diagonal terms of $\mathbf{X}(s)$ as perturbations of the diagonal terms in the case where different forcing frequencies $\omega_1 < \omega_2 < \ldots < \omega_l$ are chosen for each of the parameter tracking loops. In this case, from Corollary 4.1, we have

$$X_{pq}(s) = P_{pq} a_p L_p(s) \tag{5.55}$$

because the coupling terms $M_p(s)$ do not arise. Thus, we have

$$\frac{X_{pq}(s)}{1 + X_{pp}(s)} = \frac{P_{pq}}{P_{pp}} \frac{X_{pp}(s)}{1 + X_{pp}(s)}. \tag{5.56}$$

Taking the H_∞ norm of both sides of Eqn. (5.56), and using the submultiplicative property of the H_∞ norm, we get the following corollary to Proposition 5.1:

Theorem 5.1 *Consider the system from Theorem 4.1 with separate forcing frequencies $\omega_1 < \omega_2 < \ldots < \omega_l$ for each of the parameter tracking loops. If $\frac{X_{pp}(s)}{1+X_{pp}(s)}$ are asymptotically stable and $|P_{pq}| < \frac{\rho_l^*}{\|\frac{X_{pp}}{1+X_{pp}}\|_{H_\infty}} P_{pp}$ for each $q \neq p$, then $\frac{1}{\det(\mathbf{I}+\mathbf{X}(s))}$ is asymptotically stable.*

Hence, we can design $C_{ip}(s)$ to minimize $\|\frac{X_{pp}}{1+X_{pp}}\|_{H_\infty}$ for each p which maximizes the allowable $|\frac{P_{pq}}{P_{pp}}|$. Diagonal dominance in a positive definite matrix $\mathbf{P}$ simply means that the coordinate axes of the level surfaces $(\theta - \theta^*)^T \mathbf{P} (\theta - \theta^*)$ are close to the principal axes in orientation. The need for dominance of diagonal terms in the Hessian of the nonlinearity $\mathbf{P}$ thus has a simple geometric interpretation: the inputs θ should be close to being along the principal axes of the level surfaces of the nonlinearity. Clearly, the difficulty of control design increases with dimension as ρ_l^* decreases roughly as $1/l$. For high dimensions, the problem may not have a solution. For the important case of optimizing a static map, where $F_{ip}(s) = F_o(s) = 1$, $\Gamma_{\theta p}(s) = 1/s$ for each p, and $\Gamma_f(s) = 1/s$, we can choose separate forcing frequencies for each of the parameter tracking loops, $C_{op}(s) = 1/(s+h), h > 0$, for all p, $C_{ip}(s) = k_p > 0$, and obtain $X_{pp}(s) = \frac{k_p a_p P_{pp}(s^2+hs+\omega_p^2)}{s((s+h)^2+\omega_p^2)}$. If $k_p a_p P_{pp} < h$, we have $\|\frac{X_{pp}}{1+X_{pp}}\|_{H_\infty} = 1$ and this yields the condition for stability as $|P_{pq}| < \rho_l^* P_{pp}$ for each $q \neq p$ from Theorem 5.1. To sum up the process of design, we state a multiparameter design algorithm:

Algorithm 5.1 (Multiparameter Design Algorithm)

1. *Select $\omega_1, \omega_2, \ldots, \omega_l$ sufficiently large, not equal to frequencies in noise, and with $\pm j\omega_p$ not equal to imaginary axis zeros of $F_{ip}(s)$.*
2. *Set perturbation amplitudes a_p so as to obtain small steady state output error $\tilde{y}$.*
3. *Design each $C_{op}(s)$ asymptotically stable, with zeros that include the zeros of $\Gamma_f(s)$ that are not asymptotically stable, and such that $\frac{C_{op}(s)}{\Gamma_f(s)}$ is proper.*
4. *For each $p = 1, \ldots, l$, design $C_{ip}(s)$ such it does not include poles of $\Gamma_{\theta p}(s)$ that are not asymptotically stable as its zeros, $C_{ip}(s)\Gamma_{\theta p}(s)$ is proper, and $\frac{1}{\det(\mathbf{I}+\mathbf{X}(s))}$ is asymptotically stable. Asymptotic stability of $\frac{1}{\det(\mathbf{I}+\mathbf{X}(s))}$ may be achieved by designing $C_{ip}(s)$ to minimize $\|\frac{X_{pp}}{1+X_{pp}}\|_{H_\infty}$ for each p, using the result in Theorem 5.1.*

We note that the theory permits the forcing frequencies to be very close. Further, the condition $\omega_p + \omega_q \neq \omega_r$ for each $p, q, r = 1, \ldots, l$ used in [4] is not necessary for the output to converge to a neighbourhood of the extremum, but helpful in simplifying the analysis; it ensures that the averaged Eqn. (4.42) has its equilibrium at the origin, in the case of a quadratic nonlinearity.

We can either use a separate frequency for each parameter tracking loop or use one frequency for every two parameter tracking loops, or something in between. In general, using a single frequency to force two parameter tracking loops leads to greater coupling, and consequent difficulty of design.

Design Variations. The design procedure for multiparameter extremum seeking offers theoretical guarantees of local stability and performance. The results rest upon an averaging analysis that averages out oscillatory terms in $Y_{in}(s)[w_n(t)] = \mathrm{O}(\sum_{p=1}^{l}\sum_{q=1}^{l} a_p a_q)$ and in $Y_{in}(s)\left[\sum_p \sum_q \mathcal{T}_{npq}[\tilde{\theta}_q] - \mathcal{K}_{npq}[\tilde{\theta}_q]\right]$ (see Eqn. (4.35)). The magnitude of these oscillations can be large, and can mean a highly oscillatory output about the extremum, or even loss of stability. Here we propose design variations within the framework of the analysis above that enhance the practical utility of the design algorithm by attenuation of the oscillatory terms by a factor ϵ as they pass through the plant ($\mathbf{F}_i(s)$ and $F_o(s)$) and filters ($\frac{C_{on}(s)}{\Gamma_f(s)}$ and $C_{in}(s)\Gamma_{\theta n}(s)$):

1. Attenuation through plant dynamics, $\mathbf{F}_i(s)$ and $F_o(s)$ (High frequency design):
   (a) Select ω_1 such that $F_{in}(j\Omega) < \epsilon$ for each $n = 1, \ldots, l$, and $|F_o(j\Omega)| < \epsilon$ for all $\Omega > \omega_1$.
   (b) Choose each of the other frequencies ω_n large enough to attain $|\omega_n - \omega_p - \omega_q| \geq \omega_1$ for all $n, p, q = 1, \ldots, l$. This will yield $|F_{in}(j(\omega_n - \omega_p - \omega_q))|, |F_o(j(\omega_n - \omega_p - \omega_q))| \leq \epsilon$.

(c) Perform steps 2, 3, 4 in Algorithm 5.1.

If both $\mathbf{F}_i(s)$ and $F_o(s)$ are relative degree zero, we will not be able to achieve arbitrary attenuation ϵ.

2. Attenuation through tracking compensator $C_{in}(s)$:

   (a) Perform steps 1, 2, 3 of Algorithm 5.1 and write $C_{in}(s) = C'_{in}(s)F_{LPn}(s)$.

   (b) Design an asymptotically stable, minimum phase low-pass filter $F_{LPn}(s)$ such that $|F_{LPn}(j\Omega)| \le \epsilon$ for all $\Omega > |\omega_n - \omega_p - \omega_q|$ for all $n, p, q = 1, \ldots, l$.

   (c) Design each $C'_{in}(s)$ as the $C_{in}(s)$ in step 4 of Algorithm 5.1 with $\Gamma_{\theta n}(s)$ replaced by $F_{LPn}(s)\Gamma_{\theta n}(s)$ with the additional constraint that poles and zeros in it do not cancel any poles or zeros of $F_{LPn}(s)$.

3. Attenuation through output compensator $C_{on}(s)$:

   (a) Perform steps 1, and 2 of Algorithm 5.1 and write $C_{on}(s) = C'_{on}(s)F_{BPn}(s)$.

   (b) Design an asymptotically stable, minimum phase band-pass filter $F_{BPn}(s)$ such that $|F_{BPn}(j\Omega)| \le \epsilon$ for all $\Omega \ne \omega_n$, where $\Omega \in \{\omega_p, \omega_p \pm \omega_q, \omega_n \pm \omega_p \pm \omega_q\}$, $n, p, q = 1, \ldots, l$.

   (c) Design each $C'_{on}(s)$ as the $C_{on}(s)$ in step 3 of Algorithm 5.1 with $\Gamma_f(s)$ replaced by $\frac{\Gamma_f(s)}{F_{BPn}(s)}$, with the additional constraint that poles and zeros in it do not cancel any poles or zeros of $F_{BPn}(s)$.

   (d) Perform step 4 of Algorithm 5.1 as before.

It is clear that each one of these three design variations, whose objective is to attenuate the effect of the probing signals, will make stability harder to achieve.

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