Partial Stability of Dynamical Systems

Y. Sun, A. P. Molchanov, A. N. Michel
Department of Electrical Engineering
University of Notre Dame
South Bend, IN, 46556
USA

Abstract

Important variants and complements to the original Lyapunov and Lagrange stability concepts are corresponding notions concerning partial stability. For a given motion of a dynamical system, say $x(t, x_0, t_0) = (y(t, x_0, t_0), z(t, x_0, t_0))$, partial stability concerns the qualitative behavior of the y-component of the motion, relative to disturbances in the entire initial vector $x(t_0, x_0, t_0) = x_0 = (y_0, z_0)$, or relative to disturbances in the initial component y_0. In the former case we speak simply of *y-stability*, while in the latter case, we speak more explicitly of *y-stability* under arbitrary z-perturbations.

In the present paper we establish new results for y-stability of invariant sets and y-stability under arbitrary initial z-perturbations for dynamical systems defined on metric space, using stability preserving mappings. Our results are applicable to a much larger class of systems than existing results, including to dynamical systems that cannot be determined by the usual classical (differential) equations and inequalities. In contrast to existing results which pertain primarily to the analysis of equilibria, the present results apply to invariant sets (including equilibria as special cases). To demonstrate the applicability of the method advanced herein, we apply our results in the analysis of a class of discrete event systems (a computer load balancing problem).

1 Introduction

With the emergence of the *Second Method of Lyapunov* (resp., the *Direct Method of Lyapunov*) as an indispensable tool in science, engineering, and applied mathematics (see, e.g., [1], [2], [3], [4], [5]), several interesting and important variants and complements to Lyapunov's original concepts of stability were proposed. One of these involves the notion of *partial stability*. This type of stability is of interest, e.g., in applications where only the qualitative behavior of a prespecified component (say, component $y(t, x_0, t_0)$) of a motion (say, $x(t, x_0, t_0) = (y(t, x_0, t_0), z(t, x_0, t_0))$ is of interest or in applictions where stability with respect to only such a component (component $y(t, x_0, t_0)$) is in fact possible. The initial work in this area concerned partial stability, resp., y-stability, with respect to disturbances in the *entire* initial motion $x(t_0, x_0, t_0) = x_0 = (y_0, z_0)$ (see [6]), while in subsequent work, partial stability with respect to disturbances in only part of the initial motion, y_0, is of interest (see, e.g., [7], [8]). To simplify matters, we will refer to the former simply as *partial stability*, or *y-stability* and to the latter as *partial stability under arbitrary initial z-perturbations*, or *y-stability under arbitrary initial z-perturbations*.

The initial work concerning y-stability [6] addresses the partial stability of an equilibrium for continuous-time finite dimensional dynamical systems determined by ordinary differential equations. Subsequent results on partial stability (of an equilibrium) concern discrete-time finite dimensional dynamical systems determined by ordinary difference equations (see, e.g., [9], [10]), functional differential equations [11], systems with impulse effects (see, e.g., [12], [13], [14]), Ito differential equations [11], and others. The work in [7], [8] on y-stability of an equilibrium under arbitrary initial z-perturbations concerns dynamical systems determined by ordinary differential equations. For additional sources concerning the work described above, the reader should consult the references cited in [11]. We note in passing that problems concerning partial stability of dynamical systems are closely related to problems of *stability with respect to two measures* [15].

In this paper, to explore the y-stability and y-stability under arbitrary initial z-perturbations properties of the system under study, we utilize, as was done in earlier work [5], [16], [19], *stability preserving mappings*, i.e., mappings that preserve the stability properties of two dynamical systems. The domain of such a mapping is the *dynamical system under study* while its range is a well understood dynamical system, the *comparison system*. In this approach, the qualitative properties of the system under investigation are deduced from the qualitative properties of the comparison system. Next, we use the above results to establish the *Principal Lyapunov Theorems* of general motions for y-stability of invariant sets and y-stability under arbitrary initial z-perturbations. Finally, we analyze a class of *discrete event systems*, using some of these results (with particular application to a load balancing problem in a computer network [18]).

2 Preliminaries

Let (X, d) be a metric space, where X denotes the underlying set and d denotes the distance function. Throughout, we will find it convenient to view (X, d) as a product of two metric spaces (Y, d_y) and (Z, d_z). Then $X = Y \times Z$, i.e., for every $x \in X$, $x = (y, z)$, where $y \in Y$, $z \in Z$. The distance function d may be defined in a variety of ways, e.g., $d(x_1, x_2) = (d_y(y_1, y_2)^p + d_z(z_1, z_2)^p)^{1/p}$ for all $x_1 = (y_1, z_1)$, $x_2 = (y_2, z_2) \in X$, $1 \leq p \leq \infty$, where for $p = \infty$ we have $d(x_1, x_2) = max\{d_y(y_1, y_2), d_z(z_1, z_2)\}$. Recall that the distance between $x_0 \in X$ and a set $M \subset X$ is defined as $d(x_0, M) = inf_{\tilde{x} \in M} d(x_0, \tilde{x})$. We assume that M is the product of two sets $M_y \subset Y$ and $M_z \subset Z$, i.e., $M = M_y \times M_z$. For $x_0 = (y_0, z_0)$, we define the distance between y_0 and M_y by $d_y(y_0, M_y) = inf_{\tilde{y} \in M_y} d_y(y_0, \tilde{y})$. The distance $d_z(z_0, M_z)$ is defined similarly. Throughout, we let $T = R^+ = [0, \infty)$ or $T = N = \{0, 1, 2, ...\}$.

Definition 2.1. Let $A \subset X$. For any fixed $a \in A$, $t_0 \in T$, a mapping $p(., a, t_0) : T_{a,t_0} \to X$ is called a *motion* if $p(t_0, a, t_0) = a$, where $T_{a,t_0} = [t_0, t_1) \bigcap T$, $t_1 > t_0$ and t_1 is finite or infinite. □

Note that for $p(t, a, t_0) \in X$, we have $p(t, a, t_0) = (p_y(t, a, t_0),\ p_z(t, a, t_0))$, where $p_y(t, a, t_0) \in Y$ and $p_z(t, a, t_0) \in Z$.

Definition 2.2. Let S be a family of motions, i.e.,

$S \subset \{p(., a, t_0) \in \Lambda : p(t_0, a, t_0) = a\}$, where

$\Lambda = \bigcup_{(a,t_0)\in A\times T}\{T_{a,t_0} \times \{a\} \times \{t_0\} \rightarrow X\}$

and $T_{a,t_0} \times \{a\} \times \{t_0\} \rightarrow X$ denotes a mapping from $T_{a,t_0} \times \{a\} \times \{t_0\}$ into X. The four-tuple $\{T, X, A, S\}$ is called a *dynamical system.* □

When $T = R^+$, $\{T, X, A, S\}$ is called a continuous-time dynamical system while when $T = N$, one speaks of a *discrete-time dynamical system.* When all is clear from context, we will usually refer to a "dynamical system S", rather than a "dynamical system $\{T, X, A, S\}$".

Definition 2.3. Let $\{T, X, A, S\}$ be a dynamical system. A set $M \subset A$ is said to be *invariant* with respect to system S if $a \in M$ implies that $p(t, a, t_0) \in M$ for all $t \in T_{a,t_0}$, all $t_0 \in T$ and all $p(., a, t_0) \in S$. A set $M \subset A$ is said to be *y-invariant* with respect to system S if $a \in M$ implies that $p_y(t, a, t_0) \in M_y$ for all $t \in T_{a,t_0}$, for all $t_0 \in T$ and all $p(., a, t_0) \in S$. □

When M is invariant (resp., y-invariant) with respect to S, we will frequently say (S, M) is *invariant* (resp., *y-invariant*).

In studying partial stability, we will require the following concepts.

Definition 2.4. Let $\{T, X, A, S\}$ be a dynamical system and let $M \subset A$. We say that (S, M) is *y-stable with respect to S*, or more compactly, (S, M) is *y-stable*, if for every $\epsilon > 0$ and $t_0 \in T$ there exists a $\delta = \delta(\epsilon, t_0) > 0$ such that $d_y(p_y(t, a, t_0), M_y) < \epsilon$ for all $t \in T_{a,t_0}$ and for all $p(., a, t_0) \in S$, whenever $d(a, M) < \delta$. We say that (S, M) is *y-uniformly stable* if $\delta = \delta(\epsilon)$. We say that (S, M) is *y-attractive* if for any $t_0 \in T$ there exists an $\eta = \eta(t_0) > 0$ such that $lim_{t\rightarrow\infty} d_y(p_y(t, a, t_0), M_y) = 0$ for all $p(., a, t_0) \in S$, whenever $d(a, M) < \eta$. If (S, M) is y-stable and y-attractive, we say (S, M) is *y-asymptotically stable.* We call (S, M) *y-uniformly asymptotically stable* if (S, M) is y-uniformly stable and *y-uniformly attractive.* In this case for every $\epsilon > 0$ and every $t_0 \in T$, there exists a $\delta > 0$, independent of t_0 and ϵ, and a $\tau = \tau(\epsilon) > 0$, independent of t_0 such that $d_y(p_y(t, a, t_0), M_y) < \epsilon$ for all $t \in T_{a,t_0+\tau}$ and all $p(., a, t_0) \in S$, whenever $d(a, M) < \delta$. □

The definitions of partial stability given above are relative to the disturbances in the entire initial vector $x_0 = (y_0, z_0)$, while the definitions of partial stability under arbitrary initial z-perturbations are only relative to the disturbances in the initial component y_0 and this is the only difference between these two kinds of notions. In the following, we give the definition of y-stability under arbitrary initial z-perturbations. Other notions for y-stability under arbitrary initial z-perturbations follow from Definition 2.4, making obvious modifications.

Definition 2.5. Let $\{T, X, A, S\}$ be a dynamical system and let $M \subset A$. We say that (S, M) is *y-stable under arbitrary initial z-perturbations*, if for every $\epsilon > 0$ and $t_0 \in T$ there exists a $\delta = \delta(\epsilon, t_0) > 0$ such that $d_y(p_y(t, a, t_0), M_y) < \epsilon$ for all $t \in T_{a,t_0}$ and for all $p(., a, t_0) \in S$, whenever $d_y(a_y, M_y) < \delta$. □

The various notions of partial stability (partial stability under arbitrary initial z-perturbations) given above, are natural adaptations of the well known concepts of the corresponding types of stability of invariant sets for general dynamical systems, as discussed, e.g., in [4], [5]. Similarly as above, concepts for *y-exponential stability*, *y-asymptotic stability in the large*, *y-uniform asymptotic stability in the large*, *y-exponential stability in the large*, *y-uniform boundedness of motions* and *y-uniform ultimate boundedness* and the corresponding y-stability under arbitrary initial z-perturbations notions can also be introduced in a natural manner. We will not do this here due to space limitations. Note further that the present definitions constitute generalizations of the corresponding concepts of partial stability and partial stability under arbitrary initial z-perturbations of finite dimensional dynamical systems determined by systems of ordinary differential equations with $X = R^n$, $T = R^+$ and $M = \{0\} \subset R^n$ (refer, e.g., to [11] and the references cited therein).

3 Stability Preserving Mapping Theorem for Partial Stability under Arbitrary Initial Z-perturbations

We will utilize dynamical systems $\{T, X_1, A_1, S_1\}$ and $\{T, X_2, A_2, S_2\}$, sets $M_1 \subset A_1$ and $M_2 \subset A_2$, and a function $V : X_1 \times T \to X_2$. Define

$$\tilde{S}_2 = \mathcal{V}(S_1) \overset{\triangle}{=} \{q(.,b,t_0) : q(t,b,t_0) = V(p(t,a,t_0),t),\ p(.,a,t_0) \in S_1, t \in T,\ with\ b = V(a,t_0)\ and\ T_{b,t_0} = T_{a,t_0}, a \in A_1, t_0 \in T\}; \tag{3.1}$$

$$\tilde{M}_2 = \{x_2 \in X_2 : x_2 = V(x_1,t')\ for\ some\ x_1 \in M_1\ and\ t' \in T\};\ \text{and} \tag{3.2}$$

$$\tilde{A}_2 = \{x_2 \in X_2 : x_2 = V(x_1,t')\ for\ some\ x_1 \in A_1\ and\ t' \in T\}. \tag{3.3}$$

Note that $\mathcal{V} : S_1 \to S_2$ is induced by the mapping $V : X_1 \times T \to X_2$.

We will also require the following types of comparison functions.

Definition 3.1. A continuous function $\psi : R^+ \to R^+$ belongs to *class K* if $\psi(0) = 0$ and if ψ is strictly increasing on R^+. If ψ belongs to class K, and if $lim_{r\to\infty}\psi(r) = \infty$, then ψ belongs to *class KR*. We express this compactly as $\psi \in K$ and $\psi \in KR$, respectively. □

Theorem 3.1. Let $\{T, X_1, A_1, S_1\}$ and $\{T, X_2, A_2, S_2\}$ be two dynamical systems with $X_i = Y_i \times Z_i$, $i = 1, 2$ and let $M_1 = M_{y1} \times M_{z1} \subset A_1 \subset X_1$, $M_{y1} \subset Y_1$, $M_{z1} \subset Z_1$ and $M_2 \subset A_2 \subset X_2$. Assume there exists a function $V : X_1 \times T \to X_2$ which satisfies the following hypotheses:

i) $S_2 = \tilde{S}_2$ (see (3.1));

ii) $M_2 = \tilde{M}_2$ (see (3.2)) and $A_2 = \tilde{A}_2$ (see (3.3));

iii) there exist $\psi_1, \psi_2 \in K$ defined on R^+, such that

$$\psi_1(d_{y1}(y, M_{y1})) \le d_2(V(x,t), M_2) \le \psi_2(d_{y1}(y, M_{y1})) \tag{3.4}$$

for all $x = (y, z) \in X_1$, and $t \in T$, where d_{y1} and d_2 are the metrics on Y_1 and X_2, respectively. If M_{y1} an M_2 are closed, then the following statements are true:

(a) the invariance of (S_2, M_2) is equivalent to the *y-invariance* of (S_1, M_1), i.e., (S_2, M_2) is invariant if and only if (S_1, M_1) is y-invariant ;

(b) the stability, uniform stability, asymptotic stability and uniform asymptotic stability of (S_2, M_2) are equivalent to the *y-stability under arbitrary initial z-perturbations*, *y-uniform stability under arbitrary initial z-perturbations*, *y-asymptotic stability under arbitrary initial z-perturbations*, and *y-uniform asymptotic stability under arbitrary initial z-perturbations* of (S_1, M_1), respectively.

Proof For the definitions of stability, uniform stability, etc., refer, e.g., to [5]. Since $\mathcal{V}(S_1) = S_2$, there exists for every motion $p(., a, t_0) \in S_1$, a motion $q(., b, t_0) \in S_2$, such that $q(t, b, t_0) \triangleq V(p(t, a, t_0), t)$ for all $t \geq t_0$ with $b = V(a, t_0)$. Furthermore, for every motion $q(., b, t_0) \in S_2$, we can find a motion $p(., a, t_0) \in S_1$, such that $q(t, b, t_0) = V(p(t, a, t_0), t)$ for all $t \geq t_0$ with $b = V(a, t_0)$. Subsequently, we will use the notation $a = (a_{y1}, a_{z1}) \in A_1$, where $a_{y1} \in Y_1$, $a_{z1} \in Z_1$, and $b \in A_2$. Also, we let $p(t, a, t_0) = (p_{y1}(t, a, t_0), p_{z1}(t, a, t_0)) \in X_1$, where $p_{y1}(t, a, t_0) \in Y_1$ and $p_{z1}(t, a, t_0) \in Z_1$.

(a) We prove that *the invariance of* (S_2, M_2) *implies the y-invariance of* (S_1, M_1). Assume that (S_2, M_2) is invariant. For any $a \in M_1$, $t_0 \in T$, we have $b = V(a, t_0) \in M_2$. By the invariance of (S_2, M_2), we have that $q(t, b, t_0) \in M_2$ for all $t \in T_{b,t_0}$. From (iii), this implies $p_y(t, a, t_0) \in M_{y1}$ for all $t \in T_{a,t_0} = T_{b,t_0}$ since M_{y1} is closed. Therefore (S_1, M_1) is y-invariant.

Next, we prove that *the y-invariance of* (S_1, M_1) *implies the invariance of* (S_2, M_2). For every $b \in M_2$, $t_0 \in T$, from (ii), there is a corresponding $a \in M_1$. From the the y-invariance of (S_1, M_1), we have that $d_{y1}(p_{y1}(t, a, t_0), M_{y1}) = 0$ for all $t \in T_{a,t_0}$, and from (iii), we have that $d_2(q(t, b, t_0), M_2) = 0$ for all $t \in T_{b,t_0}$. Since M_2 is closed, we know that $q(t, b, t_0) \in M_2$. Hence (S_2, M_2) is invariant. We conclude that the invariance of (S_2, M_2) is equivalent to the y-invariance of (S_1, M_1).

(b) We first prove that *the stability of* (S_2, M_2) *implies the y-stability of* (S_1, M_1) *under arbitrary initial z-perturbations*. Assume that (S_2, M_2) is stable. Then for every $\epsilon_2 > 0$ and every $t_0 \in R^+$, there exists a $\delta_2 = \delta_2(\epsilon_2, t_0)$ such that $d_2(q(t, b, t_0), M_2) < \epsilon_2$ for all $q(., b, t_0) \in S_2$ and $t \in T_{b,t_0}$ whenever $d_2(b, M_2) < \delta_2$. We show that (S_1, M_1) is y-stable under arbitrary initial z-perturbations. For every $\epsilon_1 > 0$ and every $t_0 \in R^+$, let $\epsilon_2 = \psi_1(\epsilon_1)$ and let $\delta_1 = \psi_2^{-1}(\delta_2)$. If $d_{y1}(a_{y1}, M_{y1}) < \delta_1$, then by (iii), $d_2(b, M_2) \leq \psi_2(d_{y1}(a_{y1}, M_{y1})) < \psi_2(\delta_1) = \delta_2$. It now follows that for all $p(., a, t_0) \in S_1$ and for all $t \in T_{a,t_0} = T_{b,t_0}$, where $b = V(a, t_0)$,
$d_{y1}(p_{y1}(t, a, t_0), M_{y1}) \leq \psi_1^{-1}(d_2(V(p(t, a, t_0), t), M_2)) \leq \psi_1^{-1}(\epsilon_2) = \epsilon_1$
whenever $d_{y1}(a_{y1}, M_{y1}) < \delta_1$. Therefore, (S_1, M_1) is y-stable under arbitrary initial z-perturbations.

We now prove that *the y-stability of* (S_1, M_1) *under arbitrary initial z-perturbations implies the stability of* (S_2, M_2). Assume that (S_1, M_1) is y-stable under arbitrary initial z-perturbations. For every $\epsilon_2 > 0$ and every $t_0 \in R^+$, let $\epsilon_1 = \psi_2^{-1}(\epsilon_2)$ and let $\delta_2 = \psi_1(\delta_1)$. If $d_2(b, M_2) < \delta_2$, then $d_{y1}(a_{y1}, M_{y1}) < \psi_1^{-1}(\delta_2) = \delta_1$. From the y-stability of (S_1, M_1) under arbitrary initial z-perturbations, $d_{y1}(p_{y1}(t, a, t_0), M_{y1}) < \epsilon_1$. We therefore have $d_2(q(t, b, t_0), M_2) < \psi_2(\epsilon_1) = \epsilon_2$. Hence, (S_2, M_2) is stable.

We conclude that the stability of (S_2, M_2) is equivalent to the y-stability of (S_1, M_1) under arbitrary initial z-perturbations.

The proof that *the uniform stability of* (S_2, M_2) *is equivalent to the y-uniform stability of* (S_1, M_1) *under arbitrary initial z-perturbations* follows readily from the proof of the stability properties given above, choosing δ_1 and δ_2 to be independent of t_0.

Next, we prove that *the asymptotic stability of* (S_2, M_2) *is equivalent to the y-asymptotic stability of* (S_1, M_1) *under arbitrary initial z-perturbations.* Since we have already shown that the stability of (S_2, M_2) is equivalent to the y-stability of (S_1, M_1) under arbitrary initial z-perturbations, it suffices to prove that the attractivity of (S_2, M_2) is equivalent to the y-attractivity of (S_1, M_1) under arbitrary initial z-perturbations.

If (S_2, M_2) is attractive, there exists an $\eta_2 = \eta_2(t_0) > 0$ such that $\lim_{t\to\infty} d_2(q(t, b, t_0), M_2) = 0$ for every $q(., b, t_0) \in S_2$, whenever $d_2(b, M_2) < \eta_2$. Let $\eta_1(t_0) = \psi_2^{-1}(\eta_2(t_0))$. If $d_{y1}(a_{y1}, M_{y1}) < \eta_1$, then $d_2(b, M_2) \leq \psi_2(d_{y1}(a_{y1}, M_{y1})) < \psi_2(\eta_1) = \eta_2$, so that $\lim_{t\to\infty} d_2(q(t, b, t_0), M_2) = 0$. From (iii), $d_{y1}(p_{y1}(t, a, t_0), M_{y1}) \leq \psi_1^{-1}(d_2(q(t, b, t_0), M_2))$, so that $\lim_{t\to\infty} d_{y1}(p_{y1}(t, a, t_0), M_{y1}) = 0$. Therefore (S_1, M_1) is y-attractive under arbitrary initial z-perturbations, and hence, (S_1, M_1) is y-asymptotically stable under arbitrary initial z-perturbations.

If (S_1, M_1) is y-attractive under arbitrary initial z-perturbations, there exists an $\eta_1 = \eta_1(t_0) > 0$ such that $\lim_{t\to\infty} d_{y1}(p_{y1}(t, a, t_0), M_{y1}) = 0$ for every $p(., a, t_0) \in S_1$, whenever $d_{y1}(a_{y1}, M_{y1}) < \eta_1$. Let $\eta_2 = \psi_1(\eta_1)$. If $d_2(b, M_2) < \eta_2$, then $d_{y1}(a_{y1}, M_{y1}) \leq \psi_1^{-1}(d_2(b, M_2)) < \eta_1$, so that $\lim_{t\to\infty} d_{y1}(p_{y1}(t, a, t_0), M_{y1}) = 0$. Then $0 \leq \lim_{t\to\infty} d_2(q(t, b, t_0), M_2) \leq \lim_{t\to\infty} \psi_2(d_{y1}(p_{y1}(t, a, t_0), M_{y1})) = 0$. Hence (S_2, M_2) is attractive, and therefore asymptotically stable.

We now show that *the uniform asymptotic stability of* (S_2, M_2) *is equivalent to the y-uniform asymptotic stability of* (S_1, M_1) *under arbitrary initial z-perturbations.* We have already shown that the uniform stability of (S_2, M_2) is equivalent to the y-uniform stability of (S_1, M_1) under arbitrary initial z-perturbations. Therefore we only need to prove the uniform attractivity property.

Assume that (S_2, M_2) is uniformly attractive, i.e., for every $\epsilon_2 > 0$ and $t_0 \in R^+$, there exist a $\delta_2 > 0$ and a $\tau_2 = \tau_2(\epsilon_2) > 0$, such that $d_2(q(t, b, t_0), M_2) < \epsilon_2$ for all $t \in T_{b,t_0+\tau_2}$ whenever $d_2(b, M_2) < \delta_2$. For every $\epsilon_1 > 0$ and $t_0 \in R^+$, let $\epsilon_2 = \psi_1(\epsilon_1)$, $\delta_1 = \psi_2^{-1}(\delta_2) = \delta_1(\epsilon_1)$ and $\tau_1 = \tau_2 = \tau_1(\epsilon_1)$. When $d_{y1}(a_{y1}, M_{y1}) < \delta_1$, we have $d_2(b, M_2) \leq \psi_2(d_{y1}(a_{y1}, M_{y1})) < \psi_2(\delta_1) = \delta_2$, and thus $d_{y1}(p_{y1}(t, a, t_0), M_{y1}) \leq \psi_1^{-1}(d_2(q(t, b, t_0), M_2)) < \psi_1^{-1}(\epsilon_2) = \epsilon_1$ for every $t \in T_{a,t_0+\tau_1}$. Therefore, (S_1, M_1) is y-uniformly asymptotically stable under arbitrary initial z-perturbations.

Assume that (S_1, M_1) is y-uniformly attractive under arbitrary initial z-perturbations. For every $\epsilon_2 > 0$ and $t_0 \in R^+$, let $\epsilon_1 = \psi_2^{-1}(\epsilon_2)$, $\delta_2 = \psi_1(\delta_1) = \delta_1(\epsilon_2)$ and $\tau_2 = \tau_1 = \tau_2(\epsilon_2)$. When $d_2(b, M_2) < \delta_2$, $d_{y1}(a_{y1}, M_{y1}) \leq \psi_1^{-1}(d_2(b, M_2)) < \psi_1^{-1}(\delta_2) = \delta_1$, and thus $d_2(q(t, b, t_0), M_2) \leq \psi_2(d_{y1}(p_{y1}(t, a, t_0), M_{y1})) < \psi_2(\epsilon_1) = \epsilon_2$ for every $t \in T_{b,t_0+\tau_2}$. Therefore, (S_2, M_2) is uniformly asymptotically stable.

This completes the proof of the theorem. □

In Theorem 3.1, we utilized *stability preserving mappings* to identify dynamical systems with equivalent qualitative properties. Although this result is of great theoretical interest, its applicability is significantly limited because of the severe assumptions that $\tilde{S}_2 = S_2 = \mathcal{V}(S_1)$ and $M_2 = \tilde{M}_2$ and $A_2 = \tilde{A}_2$. However, when proving that the qualitative properties of (S_2, M_2) imply the qualitative properties of (S_1, M_1) in Theorem 3.1, we actually used the relation $\mathcal{V}(S_1) \subset S_2$ (rather than $\mathcal{V}(S_1) = S_2$) (and $M_2 \supset \tilde{M}_2$ and $A_2 \supset \tilde{A}_2$ rather than $M_2 = \tilde{M}_2$ and $A_2 = \tilde{A}_2$). This obervation yields the following easily applied *comparison theorem for y-stability under arbitrary initial z-perturbations.*

Theorem 3.2. In Theorem 3.1, replace (i) and (ii) by
(i) $S_2 \supset \mathcal{V}(S_1) = \tilde{S}_2$; and
(ii) $M_2 \supset \tilde{M}_2$ and $A_2 \supset \tilde{A}_2$.
Then the following statements are true:

(a) the invariance of (S_2, M_2) implies the *y-invariance* of (S_1, M_1);

(b) the stability, uniform stability, asymptotic stability and uniform asymptotic stability of (S_2, M_2) imply the same corresponding types of *y-stability of* (S_1, M_1) *under arbitrary initial z-perturbations.* □

4 Comparison Theorem for Partial Stability

In the case of y-stability, it is no longer possible to establish a stability preserving result which is in the spirit of Theorem 3.1. The reason for this is because of the asymmetry in assumption (iii) of Theorem 3.1 (see relation (3.4)). However, we are still able to establish an easily applied *comparison theorem* for y-stability which is in the spirit of Theorem 3.2.

Similarly as before, we will utilize dynamical systems $\{T, X_1, A_1, S_1\}$ and $\{T, X_2, A_2, S_2\}$, sets $M_1 \subset A_1$ and $M_2 \subset A_2$, related by a function $V : X_1 \times T \to X_2$, in the following manner:

(i) $S_2 \supset \mathcal{V}(S_1)$, where

$$\begin{aligned} \mathcal{V}(S_1) \;\; = \;\; & \{q(.,b,t_0) : q(t,b,t_0) = V(p(t,a,t_0),t), p(.,a,t_0) \in S_1 \; with \; b = V(a,t_0) \; and \\ & T_{b,t_0} = T_{a,t_0}, a \in A_1, t_0 \in T\}. \end{aligned} \tag{4.1}$$

Thus, $\mathcal{V} : S_1 \to S_2$ denotes the mapping of S_1 into S_2 induced by the mapping $V\colon X_1 \times T \to X_2$;

(ii) the sets M_1, M_2 and A_1, A_2 satisfy

$$M_2 \supset \{x_2 \in X_2 : x_2 = V(x_1, t') for\ some\ x_1 \in M_1\ and\ t' \in T\} \tag{4.2}$$

$$A_2 \supset \{x_2 \in X_2 : x_2 = V(x_1, t')\ for\ some\ x_1 \in A_1\ and\ t' \in T\}. \tag{4.3}$$

Theorem 4.1. Let $\{T, X_i, A_i, S_i\}$, $i = 1, 2$ be two dynamical systems with $X_i = Y_i \times Z_i$, $i = 1, 2$, and let $M_i \subset A_i \subset X_i$, $i = 1, 2$. Assume there exists a function $V : X_1 \times T \to X_2$ which satisfies the following hypotheses:

(i) $\mathcal{V}(S_1) \subset S_2$, where $\mathcal{V}(S_1)$ is defined in (4.1); also, M_1, M_2 and A_1, A_2 satisfy (4.2) and (4.3), respectively; and

(ii) there exist $\psi_1, \psi_2 \in K$, such that

$$\psi_1(d_{y1}(y, M_{y1})) \leq d_2(V(x,t), M_2) \leq \psi_2(d_1(x, M_1)) \tag{4.4}$$

for all $x = (y, z) \in X$, $t \in T$ where d_{y1}, d_1 and d_2 are the metrics on Y_1, X_1 and X_2, respectively, where $M_{y1} \subset Y_1$, $M_1 = M_{y1} \times M_{z1} \subset A_1$, and M_{y1} is closed.

Then the following statements are true:

(a) the invariance of (S_2, M_2) implies the *y-invariance* of (S_1, M_1);

(b) the stability, uniform stability, asymptotic stability and uniform asymptotic stability of (S_2, M_2) imply the same corresponding types of *y-stability* for (S_1, M_1). □

The proofs of the various parts of Theorem 4.1 are very similar to the proofs of corresponding parts of Theorem 3.1 and will therefore not be repeated here.

5 Principal Lyapunov Results

To establish Lyapunov-type theorems for partial stability of general dynamical systems we employ dynamical systems determined by scalar differential equations as comparison systems when $T = R^+$, and dynamical systems determined by scalar difference equations when $T = N$. To this end, we consider differential equations

$$Dx = g(t, x) \tag{E}$$

where $g \in C[R^+ \times R^+, R]$ (i.e., g is a continuous mapping from $R^+ \times R^+$ into R), $g(t, 0) = 0$ for all $t \in R^+$ (so that $x = 0$ is an equilibrium), and D is a fixed Dini derivative (i.e., any one of the Dini derivatives D^+, D_+, D^-, D_-). Let S_E denote the set of all solutions of (E). Then $\{T, X, A, S_E\}$ is a dynamical system with $T = R^+$, $X = A = R^+$, and $(S_E, \{0\})$ is invariant.

We also consider difference equations

$$x(k+1) = h(k, x(k)) \tag{F}$$

where $h : N \times R^+ \to R^+$ and $h(k, 0) = 0$ for all $k \in N$ (so that $x = 0$ is an equilibrium). Let S_F denote the set of all solutions of (F). Then $\{T, X, A, S_F\}$ is a dynamical system with $T = N$, $X = A = R^+$, and $(S_F, \{0\})$ is invariant.

In the following results, we still view (X, d) as a product space of (Y, d_y) and (Z, d_z) and $M = M_y \times M_z$.

Proposition 5.1. Let $\{T, X, A, S\}$ be a dynamical system, where $X = Y \times Z$, $M = M_y \times M_z \subset A$ and M_y is closed. Let $T = R^+$ or N. Assume that there exists a function $V : X \times T \to R^+$ and functions $\psi_1, \psi_2 \in K$ such that when $T = R^+$,

$$\psi_1(d_y(y, M_y)) \leq V(x, t) \leq \psi_2(d(x, M))$$

and when $T = N$,

$$\psi_1(d_y(y, M_y)) \leq V(x, k) \leq \psi_2(d(x, M))$$

for all $x \in X$ and $t \in R^+$, resp., $k \in N$.

(a) When T=R^+, if for any $p(., a, t_0) \in S$, $V(p(t, a, t_0), t)$ is continuous and nonincreasing for all $t \in T_{a,t_0}$, then (S, M) is *y-invariant* and *y-uniformly stable.*

When $T = N$, if for any $p(., a, k_0) \in S$, $V(p(k, a, k_0), k)$ is nonincreasing for all $k \in T_{a,k_0}$, then (S, M) is *y-invariant* and *y-uniformly stable.*

(b) When $T = R^+$, assume that for any $p(., a, t_0) \in S$, $V(p(t, a, t_0), t)$ is continuous and there exists a $\psi_3 \in K$ such that

$$DV(p(t, a, t_0), t) \leq -\psi_3(d(p(t, a, t_0), M))$$

for all $p(., a, t_0) \in S$, $t_0 \in R^+$ and $t \in T_{a,t_0}$.

When $T = N$, assume there exists a $\psi_3 \in K$ such that

$$DV(p(k, a, k_0), k) \overset{\triangle}{=} V(p(k+1, a, k_0), k+1) - V(p(k, a, k_0), k) \leq -\psi_3(d(p(k, a, k_0), M))$$

for all $p(., a, k_0) \in S$, $k \in T_{a,k_0}$ and $k_0 \in N$.

Then (S, M) is *y-uniformly asymptotically stable.* □

The proofs of the above results are direct consequences of Theorem 4.1, letting $S_2 = S_E$ when $T = R^+$ and $S_2 = S_F$ when $T = N$.

Similarly, we can establish the principal Lyapunov results for partial stability under arbitrary initial z-perturbations.

Proposition 5.2. Let $\{T, X, A, S\}$ be a dynamical system defined on a metric space (X, d), where $X = Y \times Z$. Let $M = M_y \times M_z \subset A \subset X$, where $M_y \subset Y$ and $M_z \subset Z$. Let $T = R^+$ or $T = N$. Assume that there exists a function $V : X \times T \to R^+$ and functions $\psi_1, \psi_2 \in K$ defined on R^+ such that when $T = R^+$,

$$\psi_1(d_y(y, M_y)) \leq V(x, t) \leq \psi_2(d_y(y, M_y)) \tag{5.1}$$

for all $x \in X$ and $t \in R^+$, and when $T = N$,

$$\psi_1(d_y(y, M_y)) \leq V(x, k) \leq \psi_2(d_y(y, M_y)) \tag{5.2}$$

for all $x \in X$ and $k \in N$.

If M_y is closed, then the following statements are true:

(a) When $T = R^+$, if for any $p(., a, t_0) \in S$, the function $V(p(t, a, t_0), t)$ is continuous and nonincreasing for all $t \in T_{a,t_0}$, then (S, M) is *y-invariant* and *y-uniformly stable under arbitrary initial z-perturbations.*

When $T = N$, if for any $p(., a, k_0) \in S$, the function $V(p(k, a, k_0), k)$ is nonincreasing for all $k \in T_{a,k_0}$, then (S, M) is *y-invariant* and *y-uniformly stable under arbitrary initial z-perturbations.*

(b) When $T = R^+$, if for any $p(., a, t_0) \in S$, $V(p(t, a, t_0), t)$ is continuous and there exists a $\psi_3 \in K$ defined on R^+ such that

$$DV(p(t, a, t_0), t) \leq -\psi_3(d(p(t, a, t_0), M))$$

for all $p(., a, t_0) \in S$, $t_0 \in R^+$ and $t \in T_{a,t_0}$, where D denotes a fixed Dini derivative with respect to $t \in R^+$, then (S, M) is *y-uniformly asymptotically stable under arbitrary initial z-perturbations.*

When $T = N$, assume that there exists a $\psi_3 \in K$ defined on R^+ such that

$$DV(p(k, a, k_0), k) \overset{\triangle}{=} V(p(k+1, a, k_0), k+1) - V(p(k, a, k_0), k) \leq -\psi_3(d(p(k, a, k_0), M))$$

for all $p(., a, k_0) \in S$, $k \in T_{a,t_0}$ and $k_0 \in N$. Then (S, M) is *y-uniformly asymptotically stable under arbitrary initial z-perturbations.* □

The proofs of the above results are direct consequences of Theorem 3.2, letting $S_2 = S_E$ when $T = R^+$ and $S_2 = S_F$ when $T = N$.

6 Applications to DES

Discrete event systems (DES) are systems whose evolution is characterized by the occurrence of events at possibly irregular time intervals. The behavior of DES can generally not be captured by conventional nonlinear discrete-time systems defined on R^n. We consider DES described by

$$G = (X, \mathcal{E}, f_e, g, E_v) \tag{6.1}$$

where X denotes the *set of states*, $\mathcal{E}$ is the *set of events*, $f_e : X \to X$ for $e \in \mathcal{E}$ are operators, $g : X \to P(\mathcal{E}) - \phi$ is the *enable function* and $E_v \subset \mathcal{E}^N$ is the *set of valid event trajectories*. (For an arbitrary set Z, Z^N denotes the set of all sequences $\{z_k\}_{k \in N}$, and $P(Z)$ denotes the power set of Z.) We require that $f_e(x)$ be defined only when $e \in g(x)$. If for some physical system it is possible that at some state no events occur, we model this by appending a *null event* e_0. When this occurs, the state remains the same while time advances. We associate "time" indices with states, $x_k \in X$, and corresponding *enabled events*, $e_k \in \mathcal{E}$, at time $k \in N$ if $e_k \in g(x_k)$. Thus, if at state $x_k \in X$, event $e_k \in \mathcal{E}$ occurs at time $k \in N$, then the next state is given by $x_{k+1} = f_{e_k}(x_k)$. Any sequence $\{x_k\} \in X^N$ such that for all k, $x_{k+1} = f_{e_k}(x_k)$, where $e_k \in g(x_k)$, is a *state trajectory*. The set of *event trajectories*, $E \subset \mathcal{E}^N$, is composed of sequences $\{e_k\} \in \mathcal{E}^N$ having the property that there exists a state trajectory $\{x_k\} \in X^N$ where for all k, $e_k \in g(x_k)$. We define the *set of valid event trajectories* $E_v \subset E \subset \mathcal{E}^N$ as those event trajectories that are *physically possible* in the DES G. We let $E_v(x_0) \subset E_v$ denote the set of all event trajectories in E_v that initiate at $x_0 \in X$. We shall also utilize a *set of allowed event trajectories*, $E_a \subset E_v$, and correspondingly, $E_a(x_0)$. All such event trajectories must be of infinite length.

Next, for fixed $k \in N$, let E_k denote an event sequence of k events that have occurred ($E_0 = \phi$, the empty sequence). If $E_k = e_0, e_1, \cdots, e_{k-1}$, let $E_k E \subset E_v(x_0)$ denote the *concatenation* of E_k and $E = e_k, e_{k+1}, \cdots$, i.e., $E_k E = e_0, e_1, \cdots, e_{k-1}, e_k, e_{k+1}, \cdots$. We let $x(x_0, E_k, k)$ denote the *state reached at time* k from $x_0 \in X$ by application of an event sequence E_k such that $E_k E \in E_v(x_0)$. By definition, $x(x_0, \phi, 0) = x_0$ for all $x_0 \in X$. We call $x(x_0, E_k, .)$ a *DES motion*. Presently, we assume that for all $x_0 \in X$, if $E_k E \in E_v(x_0)$ and $E_{k'} E' \in E_v(x(k_0, E_k, k))$, then $E_k E_{k'} E' \in E_v(x_0)$. Consequently, for all $x_0 \in X$, we have $x(x(x_0, E_k, k), E_{k'}, k') = x(x_0, E_k E_{k'}, k + k')$ for all $k, k' \in N$.

We now define S_{G,E_v} by

$$\begin{aligned} S_{G,E_v} \;=\; & \{p(., x_0, k_0) : p(k, x_0, k_0) = x(x_0, E_{k-k_0}, k - k_0),\ k \geq k_0,\ k \in N, x_0 \in X, \\ & E_{k-k_0} E \in E_v(x_0)\}. \end{aligned} \tag{6.2}$$

Let $T = N$ and $A = X$. Then $\{T, X, A, S_{G,E_v}\}$ is a dynamical system in the sense of Definition 2.2. Indeed, it is an *autonomous dynamical system* [5]. In the interests of brevity, we refer to this henceforth as a dynamical system $\{X, S_{G,E_v}\}$. We define $S_{G,E_a} \subset S_{G,E_v}$ and $\{X, S_{G,E_a}\}$ similarly.

Example (*Computer Network Load Balancing*)

Consider a network of computers specified by a diagraph, (C, A), where $C = \{1, \cdots, n\}$ represents a set of computers labeled by $i \in C$ and $A \subset C \times C$ specifies the set of connections (if $(i, j) \in A$, then computer i is connected with computer j). It is assumed that each computer has a *buffer* which holds tasks (its load) which can be processed by any of the computers in the network. Let $x_i \geq 0$ denote the load of computer $i \in C$. We also identify a *special group* of computers, $C' \subset C$, and we assume that after appropriate relabeling, we have $C' = \{1, ..., n_y < n\}$ and $\bar{C}' = \{n_y + 1, \cdots, n\}$. We assume that for each computer pair (i, j) such that $i, j \in C'$ or such that $i \in C'$ and $j \in \bar{C}'$ (resp., $i \in \bar{C}'$ and $j \in C'$), computer i (computer j) is capable of passing a portion of its load to computer j (computer i). It is also assumed that for each (i, j) with $i, j \in C'$ or with $i \in C'$, $j \in \bar{C}'$ (resp., $i \in \bar{C}', j \in C'$), computer i can sense the size of the load of computer j, and vice versa. Furthermore, it is assumed that the total load of computer group C', $\sum_{l=1}^{n_y} x_l$, is known to all affected pairs (i, j) at all time. It is also assumed that at any given time k, *only one load* may be exchanged.

We assume that initially ($k = k_0$), the distribution of loads across the computer network is uneven. In the following, we establish *Rules* for load exchange (by specifying g and f_e) which will result in a more even distribution of tasks across the computers in set C', subject to the *total load constraint*

$$0 < K_1 \leq \sum_{i \in C'} x_i \leq K_2. \tag{6.3}$$

We assume a *continuous load* model, where tasks can be subdivided arbitrarily. The *discrete load* case can be analyzed similarly. Let $X = (R^+)^n$ and let $x_k = (x_{1,k}, \cdots, x_{n,k})^T$ and $x_{k+1} = (x'_{1,k}, \cdots, x'_{n,k})^T$ denote the states at time k and $k+1$, respectively. Let $e^{ij}_{\alpha_k}$ denote the event that an amount α_k of load is passed from computer i to computer j. If the state is x_k, then for some $(i, j) \in A$, $e^{ij}_{\alpha_k}$ occurs to produce the next state x_{k+1}. Let $\mathcal{E} = \{e^{ij}_\alpha : (i, j) \in A, \alpha \in R^+\}$ denote the set of events. (Notice that e^{ij}_0 are valid events.) In the following, when we say "an event of type e^{ij}_α", we mean any event e^{ij}_α that represents the passing of an amount of load $\alpha \geq 0$, between i and j. We let $E_a \subset E_v$ denote the set of valid event trajectories having the property that events of each type e^{ij}_α occur infinitely often on each trajectory in E_a and that the *initial* load distribution does not violate the bound $\sum_{i=1}^{n_y} x_i \geq K_1$.

We will show (using Proposition 5.1) that under the *Rules* enumerated below, load balancing, as described above, is achieved.

Rules

A) Assume that at time k, constraint (6.3) is satisfied.

1) If $i, j \in C'$ and $x_i > x_j$, then $e_\alpha^{ij} \in g(x_k)$, $x_i' = x_i - \alpha$ (i.e., $x_{i,k+1} = x_{i,k} - \alpha$) and $x_j' = x_j + \alpha$, $\alpha = \frac{1}{2}|x_i - x_j|$.

2) If $i, j \in C'$ and $x_i = x_j$, then $e_0^{ij} \in g(x_k)$, $x_i' = x_i$ and $x_j' = x_j$.

3) If $i \in C'$ and $j \in \bar{C}'$, then $e_0^{ij} \in g(x_k)$, $x_i' = x_i$ and $x_j' = x_j$.

4) If $i, j \in \bar{C}'$, then $e_0^{ij} \in g(x_k)$, $x_i' = x_i$ and $x_j' = x_j$.

B) Assume that at time k, $\sum_{i=1}^{n_y} x_i > K_2$.

1) If $i, j \in C'$ and $x_i > x_j$, then Rule A-1 applies.

2) If $i, j \in C'$ and $x_i = x_j$, then Rule A-2 applies.

3) If $i \in C'$ and $j \in \bar{C}'$, we distinguish between two cases:

i) If $x_i \leq K_2/n_y$, $e_0^{ij} \in g(x_k)$. Then $x_i' = x_i$ and $x_j' = x_j$.

ii) If $x_i > K_2/n_y$, $e_\beta^{ij} \in g(x_k)$. Then $x_i' = x_i - \beta$ and $x_j' = x_j + \beta$, where $\beta = min\{\sum_{i=1}^{n_y} x_i - K_2, x_i - K_2/n_y\}$.

4) If $i, j \in \bar{C}'$, then Rule A-4 applies.

C) Assume that at time k, $\sum_{i=1}^{n_y} x_i < K_1$.

1) If $i, j \in C'$ and $x_i > x_j$, then Rule A-1 applies.

2) If $i, j \in C'$ and $x_i = x_j$, then Rule A-2 applies.

3) If $i \in C'$ and $j \in \bar{C}'$, we distinguish between two cases:

i) If $x_i \geq K_1/n_y$, $e_0^{ij} \in g(x_k)$. Then $x_i' = x_i$ and $x_j' = x_j$.

ii) If $x_i < K_1/n_y$, $e_\beta^{ji} \in g(x_k)$. Then $x_i' = x_i + \beta$ and $x_j' = x_j - \beta$, where $\beta = min\{K_1 - \sum_{i=1}^{n_y} x_i, K_2/n_y - x_i, x_j\}$.

4) If $i, j \in \bar{C}'$, then Rule A-4 applies. □

Now let $X = R^n$, $Y = R^{n_y}$, $Z = R^{n_z}$, $n = n_y + n_z$, $X = Y \times Z$, and choose $d(x^{(1)}, x^{(2)}) = \sum_{i=1}^{n} |x_i^{(1)} - x_i^{(2)}|$ for all $x^{(i)} = (y^{(i)}, z^{(i)}) \in X$, $i = 1, 2$. Let $d_y(y^{(1)}, y^{(2)}) = \sum_{i=1}^{n_y} |x_i^{(1)} - x_i^{(2)}|$. Choose

$$M = \{x = (y, z) \in (R^+)^n : x_1 = \cdots = x_{n_y} = c_y \ and \ K_1 \leq \sum_{i=1}^{n_y} x_i \leq K_2\} \tag{6.4}$$

$$M_y = \{y \in (R^+)^{n_y} : x_1 = \cdots = x_{n_y} = c_y \ and K_1 \leq \sum_{i=1}^{n_y} x_i \leq K_2\}. \tag{6.5}$$

In view of the Rules for g and f_e enumerated above, it is clear that M is invariant and y-invariant with respect to both E_v and E_a. Let

$$V(x) = d(x, M) = inf\{\sum_{i=1}^{n} |x_i - \tilde{x}_i|, \tilde{x} = (\tilde{x}_1, \cdots, \tilde{x}_n) \in M\} \tag{6.6}$$

$$d_y(y, M_y) = inf\{\sum_{i=1}^{n_y} |x_i - \tilde{x}_i|, \tilde{y} = (\tilde{x}_1, \cdots, \tilde{x}_{n_y}) \in M_y\}. \tag{6.7}$$

Then $V(x) = d(x, M) = d_y(y, M_y)$. Also, for all $x = (y, z) \in R^n$ and $y \in M_y$,

$$\psi_1(d_y(y, M_y)) \leq V(x) \leq \psi_2(d(x, M)) \tag{6.8}$$

where $\psi_1(r) = \psi_2(r) = r$, $r \in R^+$, i.e., $\psi_1, \psi_2 \in K$.

Let $M^* \subset M$ denote the set of points $\tilde{x}^* \in M$ where the infimum in (6.6) is achieved. Then for any $x \in R^n$ and $\tilde{x}^* \in M^*$,

$$V(x) = d(x, M) = \sum_{i=1}^{n_y} |x_i - \tilde{x}_i{}^*| = \sum_{i=1}^{n_y} |x_i - c_y^*|$$

where $\tilde{x}_i{}^* = c_y^*$, $i = 1, \cdots, n_y$.

For Rules A-$2, 3, 4$, B-$2, 3i, 4$ and C-$2, 3i, 4$, it is clear that $V(x(k+1)) = V(x(k))$. We can show that for Rules A-1, B-1, C-1, if $x_i > x_j \geq c_y^*$ or $x_j < x_i \leq c_y^*$, then $V(x(k+1)) \leq V(x(k))$ and when $x_i > c_y^* > x_j$, then $V(x(k+1)) < V(x(k))$. We can show that for Rule B-$3ii$, if $x_i > K_2/n_y$, then $V(x(k+1)) < V(x(k))$, and for Rule C-$3ii$, if $x_i < K_1/n_y$, then $V(x(k+1)) < V(x(k))$. We omit the details due to space limitations. Finally, we note that since in each event trajectory of E_a, *each type* of event e_α^{ij} occurs infinitely often, it follows in view of the properties of the Rules that $\lim_{k\to\infty} V(x(k)) = 0$. For a detailed proof, please refer to [20]

All conditions of Proposition 5.1 are satisfied, yielding the following results.

Corollary 6.1. For the computer network load balancing problem with continuous load obeying the Rules,

i) M is *invariant*, *y-invariant* and*y-stable* with respect to E_v; and

ii) M is *invariant* and *y-asymptotically stable* with respect to E_a. □

Similarly as in the above discussion, we can also establish results for partial stability under arbitrary initial z-perturbations.

Corollary 6.2. For the computer network load balancing problem with continuous load obeying the Rules,

i) M is *y-invariant* and *y-stable under arbitrary initial z-perturbations* with respect to E_v;

ii) M is *y-asymptotically stable under arbitrary initial z-perturbations* with respect to E_a. □

Remark 6.1 Conclusion (i) in Corollary 6.2 asserts that under the indicated assumptions and rules, arbitrarily small constraint violations and arbitrarily small initial load inbalances in computer group C' will remain arbitrarily small, even though the initial load inbalances in computer group $\bar{C}'$ may be arbitrarily large. Conclusion (ii) asserts that if there is sufficient load in computer group C' ($\sum_{i\in C'} x_i \geq K_1$) and if in every event trajectory, each type of event e_α^{ij} occurs infinitely often, the computer group C' will eventually have balanced load, satisfying the load constraint (6.3), no matter what the initial load inbalances in group $\bar{C}'$ may be. In contrast to this, in the case of Corollary 6.1, the interpertation for y-stability demands that arbitrarily small constraint violations and arbitrarily small initial load inbalances in the *entire* computer group C will result in inbalances in computer group C' that remain arbitrarily small (even though this restriction was not used in proving the present example). □

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