An Approach to General Switched Linear Quadratic Optimal Control Problems with State Jumps

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Abstract

Unlike conventional optimal control problems, optimal control problems of switched systems require the solutions of not only the optimal continuous inputs but also the optimal switching sequences. In a previous paper by the authors, an approach for an important class of switched systems optimal control problems, namely, general switched linear quadratic (GSLQ) problems where each subsystem is linear and the cost functionals are in general quadratic forms, was reported. In this paper, we extend the approach to GSLQ problems with state jumps at the switching instants. For such problems, the cost functionals include not only the general quadratic cost terms for the state and the input but also the costs for state jumps. The approach in this paper allows us to derive the derivatives of the optimal cost with respect to the switching instants based on the solution of the discontinuous Riccati equation parameterized by the switching instants along with its differentiations. With the knowledge of the derivatives, nonlinear optimization methods can be applied to locate the optimal switching instants. An example is provided to illustrate the approach.

1 Introduction

A switched system is a particular kind of hybrid system that consists of several subsystems and a switching law orchestrating the active subsystem at each time instant. The system states can be continuous or have discontinuous jumps at the switching instants. Many real-world processes such as mechanical systems, automotive systems, and electrical circuit systems, etc., can be modeled as such systems.

Optimal control problems are one of the most challenging and important classes of problems for switched systems. Unlike conventional optimal control problems, optimal control problems of switched systems require the solutions of not only the optimal continuous inputs but also the optimal switching sequences. Many literature results have appeared for problems without state discontinuities (see e.g., [6, 7, 9, 10, 11, 12]). In a previous paper [13], we proposed an approach to an important class of switched systems optimal control problems, namely, general switched linear quadratic (GSLQ) problems where each subsystem is linear and the cost functionals are in general quadratic forms. However, theoretical or practical results for optimal control of switched systems with state jumps have rarely be reported in the literature (see e.g., [2, 3, 4, 5, 8]; [3, 4] deal with autonomous switched systems and

[2, 5, 8] propose some theoretical results). In such problems, the discontinuities of the system states at the switching instants pose additional difficulties.

In this paper, we extend the approach in [13] to GSLQ problems with state jumps. Since many practical problems only involve optimization where the sequence of active subsystems are prespecified, we focus on such problems. We first carefully formulate the problem so that linear jumps and quadratic costs for switchings are taken into consideration. An algorithm is then given. In order to apply it, the derivatives of the optimal cost with respect to the switching instants need to be known. Our approach first transcribes a GSLQ problem into an equivalent conventional problem parameterized by the switching instants and then obtains the derivative values based on the solution of the discontinuous Riccati equation parameterized by the switching instants along with its differentiations.

The structure of the paper is as follows. In Section 2, we introduce the model of switched systems with state jumps and formulate the GSLQ problems. In Section 3, we review an algorithm proposed in [13]. In Section 4, we propose a method that transcribes a GSLQ problem into an equivalent conventional optimal control problem with state jumps. In Section 5, it is shown how to obtain the derivatives of the optimal cost with respect to the switching instants based on the solution of the discontinuous Riccati equation along with its differentiations. Section 6 provides an example to illustrate our approach. Section 7 concludes the paper.

2 Problem Formulation

2.1 Switched Systems

In this paper, we consider *switched linear systems* consisting of the subsystems

$$\dot{x} = A_i x + B_i u \tag{2.1}$$

where $A_i \in \mathbb{R}^{n\times n}$, $B_i \in \mathbb{R}^{n\times m}$, $i \in I \triangleq \{1, 2, \cdots, M\}$. In order to control such a switched system, one needs to choose not only a continuous input but also a switching sequence. A *switching sequence* in $t \in [t_0, t_f]$ regulates the sequence of active subsystems and can be defined as

$$\sigma = ((t_0, i_0), (t_1, i_1), \cdots, (t_K, i_K)) \tag{2.2}$$

where $0 \leq K < \infty$, $t_0 \leq t_1 \leq t_2 \leq \cdots \leq t_K \leq t_f$, $i_k \in I$ for $k = 0, 1, \cdots, K$. Note here (i_k, t_k) indicates that at instant t_k, the system switches from subsystem i_{k-1} to subsystem i_k; during the time interval $[t_k, t_{k+1})$ ($[t_K, t_f]$ if $k = K$), subsystem i_k will be active. For a switched system to be well-behaved, we only consider nonZeno sequences which switch at most a finite number of times in $[t_0, t_f]$, though different sequences may have different numbers of switchings. If we regard σ as a discrete input, then the overall control input to the system is a pair (σ, u).

Note that at the switching instants, the system state may exhibit discontinuous jumps. In the sequel, we are particularly interested in *switched linear systems with state jumps* at the switching instants. The class of state jumps at switching instant t_k considered in this paper are linear and are described by

$$x(t_k+) = \Theta_{i_k,i_{k+1}} x(t_k-) + \Gamma_{i_k,i_{k+1}} \tag{2.3}$$

where $\Theta_{i_k,i_{k+1}} \in \mathbb{R}^{n\times n}$, $\Gamma_{i_k,i_{k+1}} \in \mathbb{R}^n$, $k = 1, 2, \cdots, K$.

2.2 General Switched Linear Quadratic (GSLQ) Optimal Control Problems with State Jumps

In the sequel, we define define $\mathcal{U}_{[t_0,t_f]} \triangleq \{u | u \in C_p[t_0, t_f], u(t) \in \mathbb{R}^m\}$; in other words, $\mathcal{U}_{[t_0,t_f]}$ is the set of all piecewise continuous functions for $t \in [t_0, t_f]$ that take values in $\mathbb{R}^m$. Since many practical problems only involve optimizations in which a prespecified sequence of active subsystems (i.e., the untimed sequence $(i_0, i_1, \cdots, i_K)$) is given, we concentrate on such problems. (Such problems appear, e.g., in the speeding up of an automobile power train which only requires switchings from gear 1 to 2 to 3 to 4.)

Problem 2.1. *Consider a switched linear system with state jumps. Given a fixed time interval $[t_0, t_f]$ and given a prespecified sequence of active subsystems $(i_0, i_1, \cdots, i_K)$, find switching instants $t_1, \cdots, t_K$ and a continuous input $u \in \mathcal{U}_{[t_0,t_f]}$ such that the corresponding continuous state trajectory x departs from a given initial state $x(t_0) = x_0$ and the cost functional*

$$J = \psi(x(t_f)) + \int_{t_0}^{t_f} L(x(t), u(t))\ dt + \sum_{k=1}^{K} \psi_k(x(t_k-)) \tag{2.4}$$

where

$$\begin{aligned}
\psi(x(t_f)) &= \frac{1}{2}x^T(t_f)Q_f x(t_f) + M_f x(t_f) + W_f, && (2.5)\\
L(x, u) &= \frac{1}{2}x^T Q x + x^T V u + \frac{1}{2}u^T R u + M x + N u + W, && (2.6)\\
\psi_k(x(t_k-)) &= \frac{1}{2}x^T(t_k-)Q_{i_k,i_{k+1}} x(t_k-) + M_{i_k,i_{k+1}} x(t_k-) + W_{i_k,i_{k+1}}, && (2.7)
\end{aligned}$$

is minimized. Here t_0, t_f and $x(t_0) = x_0$ are given; Q_f,M_f,W_f,Q,V,R,M,N,W,$Q_{i_k,i_{k+1}}$'s, $M_{i_k,i_{k+1}}$'s, and $W_{i_k,i_{k+1}}$'s are matrices of appropriate dimensions with $Q_f \geq 0$, $Q \geq 0$, $Q_{i_k,i_{k+1}} \geq 0$, and $R > 0$. □

Remark 2.1. In (2.4), the term $\sum_{k=1}^{K} \psi_k(x(t_k-))$ denotes the sum of the costs incurred at the switching instants. It is without loss of generality that we denote each ψ_k as a quadratic function in $x(t_k-)$. In the case that the switching cost is a quadratic function in $(x(t_k+) - x(t_k-))$, we can utilize the relationship (2.3) to reduce the expression into a quadratic function in $x(t_k-)$. □

3 An Algorithm

In previous papers [12, 13], we proposed an idea which decomposes Problem 2.1 into two stages. Stage (a) is a conventional optimal control problem which seeks for the minimum value of J with respect to u under a given switching sequence $\sigma = ((t_0, i_0), (t_1, i_1), \cdots, (t_K, i_K))$. In the sequel, we denote the corresponding optimal cost as a function $J_1(\hat{t})$, where $\hat{t} \triangleq (t_1, t_2, \cdots, t_K)^T$. Stage (b) is a constrained nonlinear optimization problem

$$\begin{array}{c} \min_{\hat{t}} J_1(\hat{t}) \\ \text{subject to } \hat{t} \in T \end{array} \tag{3.1}$$

where $T \triangleq \{\hat{t} = (t_1, t_2, \cdots, t_K)^T | t_0 \leq t_1 \leq t_2 \leq \cdots \leq t_K \leq t_f\}$.

Based on the two stage idea, the following algorithm (see [13]) provides a framework for the optimization in the subsequent sections.

Algorithm 3.1.

(1). Set the iteration index $j = 0$. Choose an initial $\hat{t}^j$.

(2). By solving an optimal control problem (stage (a)), find $J_1(\hat{t}^j)$.

(3). Find $\frac{\partial J_1}{\partial \hat{t}}(\hat{t}^j)$ (and $\frac{\partial^2 J_1}{\partial \hat{t}^2}(\hat{t}^j)$ if second-order method is to be used).

(4). Use the gradient projection method or the constrained Newton's method to update $\hat{t}^j$ to be $\hat{t}^{j+1} = \hat{t}^j + \alpha^j d\hat{t}^j$ (here the stepsize α^j is chosen using the Armijo's rule [1]). Set the iteration index $j = j + 1$.

(5). Repeat Steps (2), (3), (4) and (5), until a prespecified termination condition is satisfied. $\square$

Note that in the above algorithm, step (2) corresponds to stage (a) and steps (3), (4) correspond to stage (b). It should be pointed out that the key elements of the above algorithm are

(a). An optimal control algorithm for step (2).

(b). The derivations of $\frac{\partial J_1}{\partial \hat{t}}$ and $\frac{\partial^2 J_1}{\partial \hat{t}^2}$ for step (3).

(c). A nonlinear optimization algorithm for step (4).

Note that (a) can be dealt with by using numerical methods for conventional optimal control problems and (c) can be dealt with by using for example feasible direction methods for constrained nonlinear optimization. However, (b) poses an obstacle because the analytical expressions of $J_1(\hat{t})$ are almost impossible to obtain except for very few classes of problems. The unavailability of analytical expressions of $J_1(\hat{t})$ hence makes the values of $\frac{\partial J_1}{\partial \hat{t}}$ and $\frac{\partial^2 J_1}{\partial \hat{t}^2}$ difficult to obtain. It is the task of the subsequent sections to address (b) and derive an approach for deriving the values of $\frac{\partial J_1}{\partial \hat{t}}$ and $\frac{\partial^2 J_1}{\partial \hat{t}^2}$.

Remark 3.1. Note that for the GSLQ Problem 2.1, the optimal cost function $J_1(\hat{t})$ can be proven to be smooth (hence we can take its derivatives). The proof can be carried out by the variational arguments. We do not include the proof here for the brevity of the paper. □

4 An Equivalent Problem Formulation

Henceforth, we develop an approach for finding the derivative values of J_1 so that Algorithm 3.1 can be applied. In this section, we transcribe a GSLQ problem with state jumps into an equivalent conventional optimal control problem with state jumps parameterized by the unknown switching instants. A specific feature of the equivalent problem is that the independent time variable has the property that the switching instants are fixed with respect to it.

For convenience of notation and clarity of the presentation of the main idea of our approach, in Sections 4 and 5, we will concentrate on the case of two subsystems where subsystem 1 is active in the interval $[t_0, t_1)$ and subsystem 2 is active in the interval $[t_1, t_f]$ (t_1 is the switching instant to be determined). The approach works similarly for more than one switchings, and at the end of Section 5 we will comment on this.

Problem 4.1. *For a switched system*

$$\begin{aligned} \dot{x} &= A_1 x + B_1 u,\ t_0 \le t < t_1, && (4.1) \\ \dot{x} &= A_2 x + B_2 u,\ t_1 < t \le t_f, && (4.2) \end{aligned}$$

with state jump

$$x(t_1+) = \Theta_{1,2} x(t_1-) + \Gamma_{1,2}, \tag{4.3}$$

find an optimal switching instant t_1 and an optimal $u(t)$ such that the cost functional

$$J = \psi(x(t_f)) + \int_{t_0}^{t_f} L(x(t), u(t))\ dt + \psi_1(x(t_1-)) \tag{4.4}$$

where ψ, L, and ψ_1 are in general quadratic forms (as in (2.5)-(2.7)) is minimized. Here t_0, t_f and $x(t_0) = x_0$. □

As in [13], we transcribe Problem 4.1 into an equivalent problem in the followings.

We introduce a new variable x_{n+1} corresponding to the switching instant t_1. Let x_{n+1} satisfy

$$\frac{dx_{n+1}}{dt} = 0,\ x_{n+1}(0) = t_1. \tag{4.5}$$

Next a new independent time variable τ is introduced. A piecewise linear correspondence relationship between t and τ is established as follows.

$$t = \begin{cases} t_0 + (x_{n+1} - t_0)\tau, & 0 \le \tau \le 1, \\ x_{n+1} + (t_f - x_{n+1})(\tau - 1), & 1 \le \tau \le 2. \end{cases} \tag{4.6}$$

Note $\tau = 0$ corresponds to $t = t_0$, $\tau = 1$ to $t = t_1$, and $\tau = 2$ to $t = t_f$. By introducing x_{n+1} and τ, Problem 4.1 can be transcribed into

Problem 4.2 (Equivalent Problem). *For a system with dynamics*

$$\frac{dx(\tau)}{d\tau} = (x_{n+1} - t_0)(A_1 x + B_1 u), \tag{4.7}$$

$$\frac{dx_{n+1}}{d\tau} = 0, \tag{4.8}$$

for $\tau \in [0, 1)$ *and*

$$\frac{dx(\tau)}{d\tau} = (t_f - x_{n+1})(A_2 x + B_2 u), \tag{4.9}$$

$$\frac{dx_{n+1}}{d\tau} = 0, \tag{4.10}$$

for $\tau \in (1, 2]$ *with state jump*

$$x(1+) = \Theta_{1,2} x(1-) + \Gamma_{1,2}, \tag{4.11}$$

find optimal x_{n+1} *and* $u(t)$ *such that the cost functional*

$$J = \psi(x(2)) + \int_0^1 (x_{n+1} - t_0) L(x(t), u(t))\ d\tau + \int_1^2 (t_f - x_{n+1}) L(x, u)\ d\tau + \psi_1(x(1-)) \tag{4.12}$$

is minimized. Here $x(0) = x_0$ *is given.* □

Remark 4.1. Problem 4.2 and 4.1 are equivalent in the sense that a solution for Problem 4.2 is also a solution for Problem 4.1 by a proper change of independent variables as in (4.6) and by regarding $x_{n+1} = t_1$, and vice versa. □

Remark 4.2. Problem 4.2 provides us with the advantage that it no longer has a varying switching instant. Actually, because x_{n+1} is actually an unknown constant throughout $\tau \in [0, 2]$, Problem 4.2 can be regarded as a conventional optimal control problem with state jumps parameterized by x_{n+1}. The problem is conventional because it has fixed time instant when the system dynamics changes. □

5 The Approach

In this section, based on the equivalent problem formulation in Section 4, we develop an approach for finding $\frac{\partial J_1}{\partial t_1}$ by studying the equivalent Problem 4.2.

As indicated in Remark 4.2, the equivalent Problem 4.2 can be regarded as a GSLQ problem with state jump discontinuity parameterized by the switching instant x_{n+1}. Assume we are given a fixed x_{n+1} and assume the optimal value function is

$$V^*(x, \tau, x_{n+1}) = \frac{1}{2} x^T P(\tau, x_{n+1}) x + S(\tau, x_{n+1}) x + T(\tau, x_{n+1}) \tag{5.1}$$

where $P^T(\tau, x_{n+1}) = P(\tau, x_{n+1})$. By using the dynamic programming approach we have the following equations

$$
\begin{aligned}
-V_\tau^* &= (x_{n+1} - t_0)[V_x^*(A_1 x + B_1 u) + L(x,u)], \text{ for } t \in [0,1) && (5.2)\\
-V_\tau^* &= (t_f - x_{n+1})[V_x^*(A_2 x + B_2 u) + L(x,u)], \text{ for } t \in (1,2] && (5.3)\\
V^*(x, 1-, x_{n+1}) &= V^*(\Theta_{1,2} x + \Gamma_{1,2}, 1+, x_{n+1}) + \psi_1(x) && (5.4)
\end{aligned}
$$

and solving the resultant HJB equation for $\tau \in [0,1)$ we can obtain the optimal control

$$u(x, \tau, x_{n+1}) = -K(\tau, x_{n+1}) x(\tau, x_{n+1}) - E(\tau, x_{n+1}) \tag{5.5}$$

where

$$
\begin{aligned}
K(\tau, x_{n+1}) &= R^{-1}[B_1^T P(\tau, x_{n+1}) + V^T], && (5.6)\\
E(\tau, x_{n+1}) &= R^{-1}[B_1^T S^T(\tau, x_{n+1}) + N^T], && (5.7)
\end{aligned}
$$

and $P(\tau, x_{n+1})$, $S(\tau, x_{n+1})$ and $T(\tau, x_{n+1})$ (abbreviated as P, S and T) satisfy the following parameterized general Riccati equation (parameterized by x_{n+1})

$$
\begin{aligned}
-\frac{\partial P}{\partial \tau} &= (x_{n+1} - t_0)[Q + PA_1 + A_1^T P - (PB_1 + V)R^{-1}(B_1^T P + V^T)], && (5.8)\\
-\frac{\partial S}{\partial \tau} &= (x_{n+1} - t_0)[M + SA_1 - (N + SB_1)R^{-1}(B_1^T P + V^T)], && (5.9)\\
-\frac{\partial T}{\partial \tau} &= (x_{n+1} - t_0)[W - \frac{1}{2}(N + SB_1)R^{-1}(B_1^T S^T + N^T)]. && (5.10)
\end{aligned}
$$

The optimal control for $\tau \in (1,2]$ is

$$u(x, \tau, x_{n+1}) = -K(\tau, x_{n+1}) x(\tau, x_{n+1}) - E(\tau, x_{n+1}) \tag{5.11}$$

where

$$
\begin{aligned}
K(\tau, x_{n+1}) &= R^{-1}[B_2^T P(\tau, x_{n+1}) + V^T], && (5.12)\\
E(\tau, x_{n+1}) &= R^{-1}[B_2^T S^T(\tau, x_{n+1}) + N^T], && (5.13)
\end{aligned}
$$

and P, S and T satisfies the following parameterized general Riccati equation

$$
\begin{aligned}
-\frac{\partial P}{\partial \tau} &= (t_f - x_{n+1})[Q + PA_2 + A_2^T P - (PB_2 + V)R^{-1}(B_2^T P + V^T)], && (5.14)\\
-\frac{\partial S}{\partial \tau} &= (t_f - x_{n+1})[M + SA_2 - (N + SB_2)R^{-1}(B_2^T P + V^T)], && (5.15)\\
-\frac{\partial T}{\partial \tau} &= (t_f - x_{n+1})[W - \frac{1}{2}(N + SB_2)R^{-1}(B_2^T S^T + N^T)]. && (5.16)
\end{aligned}
$$

At the switching instant $\tau = 1$, from (5.4), we have discontinuous jumps for P, S and T as

$$
\begin{aligned}
P(1-, x_{n+1}) &= \Theta_{1,2}^T P(1+, x_{n+1}) \Theta_{1,2} + Q_{1,2}, && (5.17)\\
S(1-, x_{n+1}) &= \Gamma_{1,2}^T P(1+, x_{n+1}) \Theta_{1,2} + S(1+, x_{n+1}) \Theta_{1,2} + M_{1,2}, && (5.18)\\
T(1-, x_{n+1}) &= 0.5 \Gamma_{1,2}^T P(1+, x_{n+1}) \Gamma_{1,2} + S(1+, x_{n+1}) \Gamma_{1,2} + T(1+, x_{n+1}) + W_{1,2}. && (5.19)
\end{aligned}
$$

Note that the equations (5.8-5.10) and (5.14-5.16) along with the discontinuity conditions (5.17-5.19) for a discontinuous Riccati equation (for a fixed x_{n+1}). By solving it, we can obtain the parameterized optimal cost at $\tau = 0$, i.e., the optimal J_1 under fixed x_{n+1} as

$$J_1(t_1) = J_1(x_{n+1}) = V^*(x_0, 0, x_{n+1}) = \frac{1}{2}x_0^T P(0, x_{n+1})x_0 + S(0, x_{n+1})x_0 + T(0, x_{n+1}). \quad (5.20)$$

From (5.20), we have

$$\frac{dJ_1}{dx_{n+1}}(x_{n+1}) = \frac{\partial V^*}{\partial x_{n+1}}(x_0, 0, x_{n+1}) = \frac{1}{2}x_0^T \frac{\partial P}{\partial x_{n+1}}(0, x_{n+1})x_0 + \frac{\partial S}{\partial x_{n+1}}(0, x_{n+1})x_0 + \frac{\partial T}{\partial x_{n+1}}(0, x_{n+1}). \quad (5.21)$$

In order to obtain the value of $\frac{dJ_1}{dx_{n+1}}$ from (5.21), we need to know $\frac{\partial P}{\partial x_{n+1}}$, $\frac{\partial S}{\partial x_{n+1}}$ and $\frac{\partial T}{\partial x_{n+1}}$ at $(0, x_{n+1})$. To obtain these values, we differentiate (5.8-5.10) and (5.14-5.16) with respect to x_{n+1} to obtain

$$\begin{aligned}
-\frac{\partial}{\partial \tau}(\frac{\partial P}{\partial x_{n+1}}) &= [Q + PA_1 + A_1^T P - (PB_1 + V)R^{-1}(B_1^T P + V^T)] + (x_{n+1} - t_0)[\frac{\partial P}{\partial x_{n+1}}A_1 + \\
&\quad A_1^T \frac{\partial P}{\partial x_{n+1}} - (\frac{\partial P}{\partial x_{n+1}}B_1)R^{-1}(B_1^T P + V^T)) - (PB_1 + V)R^{-1}(B_1^T \frac{\partial P}{\partial x_{n+1}})], \quad (5.22) \\
-\frac{\partial}{\partial \tau}(\frac{\partial S}{\partial x_{n+1}}) &= [M + SA_1 - (N + SB_1)R^{-1}(B_1^T P + V^T)] + (x_{n+1} - t_0)[\frac{\partial S}{\partial x_{n+1}}A_1 \\
&\quad -(\frac{\partial S}{\partial x_{n+1}}B_1)R^{-1}(B_1^T P + V^T) - (N + SB_1)R^{-1}(B_1^T \frac{\partial P}{\partial x_{n+1}})], \quad (5.23) \\
-\frac{\partial}{\partial \tau}(\frac{\partial T}{\partial x_{n+1}}) &= [W - \frac{1}{2}(N + SB_1)R^{-1}(B_1^T S^T + N^T)] + (x_{n+1} - t_0)[-\frac{1}{2}(\frac{\partial S}{\partial x_{n+1}}B_1)R^{-1}(B_1^T S^T \\
&\quad +N^T) - \frac{1}{2}(N + SB_1)R^{-1}(B_1^T(\frac{\partial S}{\partial x_{n+1}})^T)], \quad (5.24)
\end{aligned}$$

for $\tau \in [0, 1)$ and

$$\begin{aligned}
-\frac{\partial}{\partial \tau}(\frac{\partial P}{\partial x_{n+1}}) &= -[Q + PA_2 + A_2^T P - (PB_2 + V)R^{-1}(B_2^T P + V^T)] + (t_f - x_{n+1})[\frac{\partial P}{\partial x_{n+1}}A_2 + \\
&\quad A_2^T \frac{\partial P}{\partial x_{n+1}} - (\frac{\partial P}{\partial x_{n+1}}B_2)R^{-1}(B_2^T P + V^T)) - (PB_2 + V)R^{-1}(B_2^T \frac{\partial P}{\partial x_{n+1}})], \quad (5.25) \\
-\frac{\partial}{\partial \tau}(\frac{\partial S}{\partial x_{n+1}}) &= -[M + SA_2 - (N + SB_2)R^{-1}(B_2^T P + V^T)] + (t_f - x_{n+1})[\frac{\partial S}{\partial x_{n+1}}A_2 \\
&\quad -(\frac{\partial S}{\partial x_{n+1}}B_2)R^{-1}(B_2^T P + V^T) - (N + SB_2)R^{-1}(B_2^T \frac{\partial P}{\partial x_{n+1}})], \quad (5.26) \\
-\frac{\partial}{\partial \tau}(\frac{\partial T}{\partial x_{n+1}}) &= -[W - \frac{1}{2}(N + SB_2)R^{-1}(B_2^T S^T + N^T)] + (t_f - x_{n+1})[-\frac{1}{2}(\frac{\partial S}{\partial x_{n+1}}B_2)R^{-1}(B_2^T S^T \\
&\quad +N^T) - \frac{1}{2}(N + SB_2)R^{-1}(B_2^T(\frac{\partial S}{\partial x_{n+1}})^T)], \quad (5.27)
\end{aligned}$$

for $\tau \in (1, 2]$.

We also differentiate the discontinuity conditions (5.17-5.19) with respect to x_{n+1} to obtain the conditions

$$\begin{aligned}
\frac{\partial P}{\partial x_{n+1}}(1-,x_{n+1}) &= \Theta_{1,2}^T\frac{\partial P}{\partial x_{n+1}}(1+,x_{n+1})\Theta_{1,2}, && (5.28)\\
\frac{\partial S}{\partial x_{n+1}}(1-,x_{n+1}) &= \Gamma_{1,2}^T\frac{\partial P}{\partial x_{n+1}}(1+,x_{n+1})\Theta_{1,2}+\frac{\partial S}{\partial x_{n+1}}(1+,x_{n+1})\Theta_{1,2}, && (5.29)\\
\frac{\partial T}{\partial x_{n+1}}(1-,x_{n+1}) &= 0.5\Gamma_{1,2}^T\frac{\partial P}{\partial x_{n+1}}(1+,x_{n+1})\Gamma_{1,2}+\frac{\partial S}{\partial x_{n+1}}(1+,x_{n+1})\Gamma_{1,2}\\
&\quad+\frac{\partial T}{\partial x_{n+1}}(1+,x_{n+1}). && (5.30)
\end{aligned}$$

Now that we have the discontinuous Riccati equation parameterized by x_{n+1} formed by (5.8-5.10), (5.14-5.16) and (5.17-5.19), along with its differentiation (5.22-5.24), (5.25-5.27) and (5.28-5.30), we can solve these equations together with the following boundary conditions at $\tau = 2$

$$\begin{array}{rr}
P(2,x_{n+1})=Q_f, & S(2,x_{n+1})=M_f,\\
T(2,x_{n+1})=W_f, & \frac{\partial P}{\partial x_{n+1}}(2,x_{n+1})=0,\\
\frac{\partial S}{\partial x_{n+1}}(2,x_{n+1})=0, & \frac{\partial T}{\partial x_{n+1}}(2,x_{n+1})=0,
\end{array} \quad (5.31)$$

form ordinary differential equation with discontinuities for P, S, T, $\frac{\partial P}{\partial x_{n+1}}$, $\frac{\partial S}{\partial x_{n+1}}$ and $\frac{\partial T}{\partial x_{n+1}}$ which can be solved efficiently using the function `ode45` in MATLAB. From the solution of this differential equation, values of $\frac{\partial P}{\partial x_{n+1}}$, $\frac{\partial S}{\partial x_{n+1}}$ and $\frac{\partial T}{\partial x_{n+1}}$ at $(0, x_{n+1})$ can be obtained and substituted into (5.21) to obtain the value of $\frac{dJ_1}{dt_1}$. Algorithm 3.1 can then be applied.

Remark 5.1. (Several Subsystems and More Than One Switchings) For GSLQ problems with state jumps consisting of K subsystems and more than one switchings, we can similarly transcribe the problem into an equivalent problem in $\tau \in [0, K+1]$. It is then straightforward to differentiate the discontinuous Riccati equation parameterized by $x_{n+1},\cdots,x_{n+K}$ (i.e., $t_1,\cdots,t_K$) to obtain additional differential equations for $\frac{\partial P}{\partial x_{n+k}}$'s, $\frac{\partial S}{\partial x_{n+k}}$'s and $\frac{\partial T}{\partial x_{n+k}}$'s. Along with the boundary conditions $P = Q_f$, $S = M_f$, $T = W_f$, $\frac{\partial P}{\partial x_{n+k}} = 0$, $\frac{\partial S}{\partial x_{n+k}} = 0$ and $\frac{\partial T}{\partial x_{n+k}} = 0$ all at $(K+1, x_{n+1}, \cdots, x_{n+K})$ for all $1 \le k \le K$, we can solve the resultant discontinuous Riccati equation along with its differentiations backwards in τ to find their values at $\tau = 0$. Once we have their values at $\tau = 0$, we can substitute them into

$$\begin{aligned}
\frac{\partial J_1}{\partial x_{n+k}} &= \frac{\partial V^*}{\partial x_{n+k}}(x_0,0,x_{n+1},\cdots,x_{n+K})\\
&= \frac{1}{2}x_0^T\frac{\partial P}{\partial x_{n+k}}(0,x_{n+1},\cdots,x_{n+k})x_0+\frac{\partial S}{\partial x_{n+k}}(0,x_{n+1},\cdots,x_{n+k})x_0\\
&\quad+\frac{\partial T}{\partial x_{n+k}}(0,x_{n+1},\cdots,x_{n+k}) && (5.32)
\end{aligned}$$

to derive the accurate values of $\frac{\partial J_1}{\partial t_k}$'s. □

Remark 5.2. (Second Order Derivatives) If we take second order partial derivatives of equation (5.20), we obtain

$$\begin{aligned}\frac{d^2J_1}{dx_{n+1}^2}(t_1) &= \frac{\partial^2 V^*}{\partial x_{n+1}^2}(x_0,0,x_{n+1}) = \frac{1}{2}x_0^T\frac{\partial^2 P}{\partial x_{n+1}^2}(0,x_{n+1})x_0 \\ &+\frac{\partial^2 S}{\partial x_{n+1}^2}(0,x_{n+1})x_0 + \frac{\partial^2 T}{\partial x_{n+1}^2}(0,x_{n+1}). \end{aligned} \tag{5.33}$$

Following similar ideas of differentiation of the parameterized discontinuous Riccati equation, we can take first and second-order differentiations of (5.8-5.10), (5.14-5.16) and (5.17-5.19) with respect to x_{n+1} and form a set of differential equations with discontinuities. Along with the initial conditions (5.31) and 0's at $\tau = 2$ for $\frac{\partial^2 P}{\partial x_{n+1}^2}$, $\frac{\partial^2 S}{\partial x_{n+1}^2}$ and $\frac{\partial^2 T}{\partial x_{n+1}^2}$, the resultant differential equation for P, S, T, $\frac{\partial P}{\partial x_{n+1}}$, $\frac{\partial S}{\partial x_{n+1}}$, $\frac{\partial T}{\partial x_{n+1}}$, $\frac{\partial^2 P}{\partial x_{n+1}^2}$, $\frac{\partial^2 S}{\partial x_{n+1}^2}$ and $\frac{\partial^2 T}{\partial x_{n+1}^2}$ can be solved and hence the accurate value of $\frac{d^2J_1}{dx_{n+1}^2}$ can be obtained and the constrained Newton's method can be applied in step (4) in Algorithm 3.1. □

6 An Example

The following example illustrate the approach developed in the previous section.

Example 6.1. Consider a switched linear system consisting of

$$\text{subsystem 1: } \dot{x} = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u, \tag{6.1}$$

$$\text{subsystem 2: } \dot{x} = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u. \tag{6.2}$$

Assume that $t_0 = 0$, $t_f = 2$ and the system switches once at $t = t_1$ $(0 \le t_1 \le 2)$ from subsystem 1 to 2. Also assume that when the system switches from subsystem 1 to 2, the system state has discontinuous jump

$$x(t_1+) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} x(t_1-) + \begin{bmatrix} 0.2 \\ 0.2 \end{bmatrix}. \tag{6.3}$$

We want to find optimal t_1 and u such that

$$J = \frac{1}{2}(x_1(2)-1)^2 + \frac{1}{2}(x_2(2)+4)^2 + \frac{1}{2}\int_0^2 u^2(t)\,dt + \frac{1}{2}x_2^2(t_1-)$$

is minimized. Here $x(0) = [4,\ 4]^T$.

We use the approach in this paper to obtain the value of $\frac{dJ_1}{dt_1}$. From an initial nominal $t_1 = 1.5$, by using Algorithm 3.1 with the gradient projection method, after 9 iterations we find that the optimal switching instant is $t_1 = 0.6375$ and the corresponding optimal cost is 10.7165. The corresponding optimal continuous control and state trajectory are shown in Figure 1 (a) and (b), respectively. Figure 2 shows the optimal cost for different t_1's. □

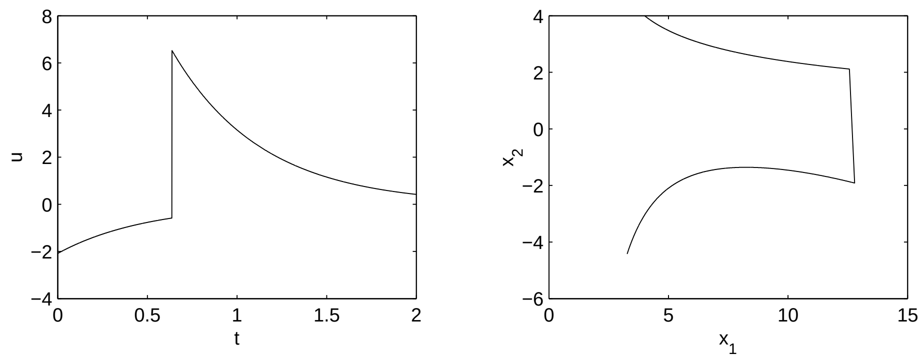

Figure 1: Example 6.1: (a) The control input. (b) The state trajectory.

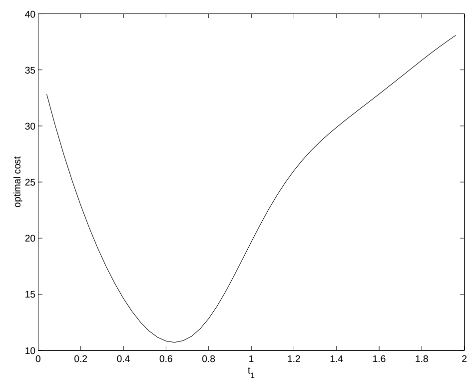

Figure 2: The optimal cost for Example 6.1 for different t_1's.

7 Conclusion

In this paper, an approach for solving GSLQ optimal control problems with state jumps is proposed. The approach is an extension of our previous approach to GSLQ problems without jumps. The approach is based on solving the discontinuous Riccati equation parameterized by the switching instants and its differentiations. Derivatives of the optimal cost with respect to the switching instants can be obtained accurately, therefore nonlinear optimization algorithms can be used to find the optimal switching instants. We believe that the approach proposed here is new and is among the very few results that address problems with state jumps. Further research topics include the study of optimal control problems of general switched systems with state jumps.

Acknowledgement: The research in this paper is supported by the National Science Foundation (NSF ECS99-12458 & CCR01-13131) and the DARPA/ITO-NEST Program (AF-F30602-01-2-0526).

References

[1] D. P. Bertsekas, "Nonlinear Programming," Second Edition, Athena Scientific, 1999.

[2] M. S. Branicky, V. S. Borkar, S. K. Mitter, "A unified framework for hybrid control: model and optimal control theory," *IEEE Trans. on Auto. Contr.* **43**(1), pp. 31-45, January, 1998.

[3] A. Giua, C. Seatzu, C. Van Der Mee, "Optimal control of autonomous linear systems switched with a pre-assigned finite sequence," *Proc. of the 2001 IEEE International Symposium on Intelligent Control*, pp. 144-149, 2001.

[4] A. Giua, C. Seatzu, C. Van Der Mee, "Optimal control of switched autonomous linear systems," *Proc. of the 40th IEEE CDC*, pp. 2472-2477, 2001.

[5] K. Gokbayrak, C. G. Cassandras, "A hierarchical decomposition method for optimal control of hybrid systems," *Proc. of the 39th IEEE CDC*, pp. 1816-1821, 2000.

[6] S. Hedlund, A. Rantzer, "Optimal control of hybrid systems," *Proc. of the 38th IEEE CDC*, pp. 3972-3977, 1999.

[7] P. Riedinger, C. Zanne, F. Kratz, "Time optimal control of hybrid systems," *Proc. of the 1999 ACC*, pp. 2466-2470, 1999.

[8] P. Riedinger, F. Kratz, C. Iung, C. Zanne, "Linear quadratic optimization for hybrid systems," *Proc. of the 38th IEEE CDC*, pp. 3059-3064, 1999.

[9] H. Sussmann, "A maximum principle for hybrid optimal control problems," *Proc. of the 38th IEEE CDC*, 1999.

[10] L. Y. Wang, A. Beydoun, J. Cook, J. Sun, I. Kolmanovsky, "Optimal hybrid control with applications to automotive powertrain systems," *LNCIS 222*, Springer-Verlag, 1997.

[11] H. S. Witsenhausen, "A class of hybrid-state continuous-time dynamic systems," *IEEE Trans. on Auto. Contr.* **11**(2), pp. 161-167, April, 1966.

[12] X. Xu, P. J. Antsaklis, "Optimal control of switched systems: new results and open problems," *Proc. of the 2000 ACC*, pp. 2683-2687, 2000.

[13] X. Xu, P. J. Antsaklis, "An approach for solving general switched linear quadratic optimal control problems," *Proc. of the 39th IEEE CDC*, pp. 1822-1827, 2000.