# Geometrical and Spectral Properties of the Time-Varying Riccati Difference Equation

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**Abstract**

In this paper discrete time-varying systems are considered. The study of spectral properties in a time-varying framework is performed by defining a suitable operator on the Hilbert space of square summable sequences. This setting gives a parameterization of square summable symmetric solutions for time-varying Riccati equations.

## 1 Introduction

It is well-known that finite dimensional Linear Time-Varying (LTV) state space models are encompassed on a operator-theoretic setting. With this approach spectral tests are available to test reachability and observability of the underlying time-varying system [4]. Spectral theory is well-established for compact operators so we will assume that the LTV system under study gives rise to a compact operator in $\ell_2^n$, a separable Hilbert space. To encompass a time-varying system $y(t) = A(t)x(t),\ x \in \mathbb{R}^n$ on the Banach space $\mathcal{L}(X, X)$ of continuos linear operators from $X$ to $X$, provided the separable Banach space $X := \ell_p^n, 1 \leq p < \infty$, define $A := diag_{t\in\mathbb{Z}}\{A(t)\}$, $x \mapsto y = Ax$ (see [4]). The diagonal form of $A$ tell us that is a *memoryless* system. We are interested to the time-varying system model $x(t) = A(t)x(t-1)$ and we will consider the corresponding operator $A$. We restrict ourselves to the Hilbert space $X = \ell_2^n$. The problem of finding invariant subspaces $\mathcal{T}$ of an operator $A$ ($A\mathcal{T} \subseteq \mathcal{T}$) is strictly related to the eigenvalues in the spectrum $\sigma(A)$. The fact that $A$ is memoryless together with separability simplifies the problem of finding invariant subspaces, in fact, in this case we can ensure the existence of a base in $\ell_2^n$ which is composed by invariant subspaces of $A$. Considering the space of continuos linear operators from $\ell_2^m$ to $\ell_2^p$: $\mathcal{L}(\ell_2^m, \ell_2^p)$ the *resolvent set* of $A \in \mathcal{L}(\ell_2^m, \ell_2^p)$ $\rho(A)$ is the set $\rho(A) := \{\lambda \in \mathbb{C} \mid \lambda I - A \text{ has bounded inverse}\}$, the *spectrum* is $\sigma(A) := \mathbb{C} \setminus \rho(A)$ and the *point spectrum* $\sigma_P(A)$ is the subset of $\sigma(A)$ for which no inverse of $\lambda I - A$ exists, i.e the set of *eigenvalues* of $A$, for $\lambda \in \sigma_P(A)$ we define $E_\lambda(A) := \cup_{k\in\mathbb{N}} Ker(\lambda I - A)^k$. Moreover, the *continuos spectrum* $\sigma_c(A)$ and the *residual spectrum* $\sigma_r(A)$ are the set of $\lambda$ such that $\lambda I - A$ is injective and non surjective with dense range or injective with non dense range respectively. Finally, the set of *almost eigenvalues* $\sigma_a(A)$ is the subset of $\sigma(A)$ such that there exists a sequence $\{x_n\} \in \ell_2^n$ (the almost eigenvectors)such that

$\lim_{n\to\infty} \|Ax_n - \lambda x_n\| = 0$ with $\|x_n\| = 1$. We have $\sigma(A) = \sigma_p(A) \cup \sigma_c(A) \cup \sigma_r(A)$ moreover $\sigma_p(A) \cup \sigma_c(A) \subset \sigma_a(A)$ (see [1]).

The spectral tests on the operators connected with linear discrete-time-varying state space systems which characterize reachability and observability can be confined to the set of almost eigenvalues [4]. Almost eigenvalues are well suited for our goal, since we want to analyze the geometry of the time-varying Riccati difference equation(RDE), in particular, the problem of parameterization of solutions. In this paper we will show that if we work in a $\ell_2$ setting with $\ell_2$-bounded matrices it is possible to give such a parameterization in the form of theorem 3.1.

We introduce the following definitions and notations. The *uniform exponential stability* (UAS) of $z(t) = A(t)z(t-1)$ i.e. the fact that $c > 0$ and $0 \le \beta < 1$ exist such that $\forall t \in \mathbb{Z}$ and $h \ge 0$ $\|A(t+h)A(t+h-1)\cdots A(t)\| \le c\beta^{h-1}$ is equivalent to $\lim_{n\to\infty} \|A^n\|^{1/n} < 1$ (see [3]). The adjoint of an operator $F : \ell_2^n \mapsto \ell_2^m$ is the unique linear operator $F' : \ell_2^m \mapsto \ell_2^n$ such that $\langle x, Fy\rangle_{\ell_2^m} = \langle F'x, y\rangle_{\ell_2^n}$. The indefinite inner product $J$ on $\ell_2^{2n}$ is defined, provided a direct sum decomposition $\ell_2^{2n} = \ell_2^n \dot{+} \ell_2^n$, by the formula $J(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}) = \langle x_1, y_2\rangle - \langle x_2, y_1\rangle$, $x_i, y_i \in \ell_2^n$. A subspace is called *Lagrangian* if $J(x,y) = 0$ $\forall x, y \in S$. We will write $S = span \begin{bmatrix} X \\ Y \end{bmatrix}$ if needed to denote the direct sum decomposition of $S \subseteq \ell_2^{2n}$.

We consider the following RDE

$$X(t\text{-}1) = A(t)'X(t)A(t)\text{-}A(t)'X(t)B(t)(R(t){+}B'(t)X(t)B(t))^{-1}B(t)'X(t)A(t){+}Q(t). \quad (1.1)$$

We assume $t \in \mathbb{Z}$ and bounded with respect to the $\ell_2$-norm operators $A, Q \in \mathcal{L}(\ell_2^n, \ell_2^n)$; $B \in \mathcal{L}(\ell_2^m, \ell_2^n)$; $R \in \mathcal{L}(\ell_2^m, \ell_2^m)$; moreover $R' = R$; $\langle Rx, x\rangle > \mu\|x\|_2^2$ $\forall x$, $\mu > 0$; $Q' = Q \ge 0$, finally we define $G(t) := B(t)R(t)^{-1}B(t)'$. (it is understood that $A$, $B, \ldots$ denote the diagonal infinite dimensional operators associated to the time-varying matrices $A(t)$, $B(t), \ldots$). Together with the operator $A$ another operator is used in this paper $C\sigma^{-1} - D$ ($\sigma$ is the unit forward shift). As $A$, $C\sigma^{-1} - D$ has an infinite matrix block representation on $\ell_2^{2n}$, the operator comes from the time-varying system equations $D(t)z(t) = C(t)z(t-1)$ where

$$C(t) := \begin{bmatrix} A(t) & 0 \\ -Q(t) & I \end{bmatrix} \qquad D(t) := \begin{bmatrix} I & G(t) \\ 0 & A(t)' \end{bmatrix}; \quad (1.2)$$

are recognized as the matrices of the time-varying symplectic pencil associated to RDE. It naturally arises that it is relevant for our purpose to study the spectrum of such an operator and to find how is related with $\sigma(A)$. If $X = X'$ with $\|X\|_{\ell_2} < \infty$ solution (from $-\infty$ to $+\infty$) of (1.1) then:

$$C(t) = \begin{bmatrix} I{+}G(t)X(t) & 0 \\ A(t)'X(t) & I \end{bmatrix} \begin{bmatrix} A_X(t) & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ \text{-}X(t\text{-}1) & I \end{bmatrix};$$

$$D(t) = \begin{bmatrix} I{+}G(t)X(t) & 0 \\ A(t)'X(t) & I \end{bmatrix} \begin{bmatrix} I & \bar{G}(t) \\ 0 & A_X(t)' \end{bmatrix} \begin{bmatrix} I & 0 \\ \text{-}X(t) & I \end{bmatrix}. \quad (1.3)$$

where $A_X(t):=(I+G(t)X(t))^{-1}A(t)$, $\bar{G}(t):=B(t)\bar{G}_2(t)^{-1}B(t)'$ with $\bar{G}_2(t) = R(t)+B(t)'X(t)$ $B(t)$. Moreover the following equation holds for all $t \in \mathbb{Z}$.

$$C(t)\begin{bmatrix} I \\ X(t-1) \end{bmatrix} \quad = \quad D(t)\begin{bmatrix} I \\ X(t) \end{bmatrix} A_X(t); \tag{1.4}$$

As explained before $C$ and $D$ define an infinite dimensional systems in the following way

$$(D - C\sigma^{-1})z(\cdot) = 0. \tag{1.5}$$

As in the ordinary case we define the resolvent set as $\rho(C\sigma^{-1}, D) := \{\lambda \in \mathbb{C} \cup \{\infty\} \mid C\sigma^{-1} - \lambda D \text{ has inverse}\}$ and obviously the spectrum $\sigma(C\sigma^{-1}, D) := \mathbb{C} \cup \{\infty\} \setminus \rho(C\sigma^{-1}, D)$. So the point spectrum $\sigma_P(C\sigma^{-1}, D)$ are the $\lambda$ for which $C(\cdot)\sigma^{-1} - \lambda D(\cdot)$ is non injective, we can define $E_\lambda(C\sigma^{-1}, D) := \cup_{k\in\mathbb{N}} Ker(C\sigma^{-1} - \lambda D)^k$, $\lambda \in \sigma_p(C\sigma^{-1}, D)$.

## 2 RDE and the eigenspaces of $C\sigma^{-1} - D$.

Given a $\ell_2$ solution of RDE (1.1) then we have a Lagrangian subspace of $\ell_2^{2n}$ given by $\dot{+}_{t\in\mathbb{Z}} span \begin{bmatrix} I \\ X(t) \end{bmatrix}$ From (1.3) we obtain

$$\begin{aligned} &\begin{bmatrix} (I+G(t)X(t))^{-1} & 0 \\ -A(t)'X(t)(I+G(t)X(t))^{-1} & I \end{bmatrix}(C(t)\sigma^{-1}-D(t))= \\ &\left(\begin{bmatrix} A_X(t) & 0 \\ 0 & I \end{bmatrix}\sigma^{-1}-\begin{bmatrix} I & \bar{G}(t) \\ 0 & A_X(t)' \end{bmatrix}\right)\begin{bmatrix} I & 0 \\ -X(t) & I \end{bmatrix}. \end{aligned} \tag{2.6}$$

If $X(\cdot)$ is in $\ell_2$ then $(I + G(t)X(t))^{-1}$, $A_X(t)$, and $\bar{G}(t)$ are in $\ell_2$. From the above equation we see that $\sigma(C\sigma^{-1}, D) = \sigma\left(\begin{bmatrix} A_X & 0 \\ 0 & I \end{bmatrix}\sigma^{-1}, \begin{bmatrix} I & \bar{G} \\ 0 & A_X' \end{bmatrix}\right)$ and it can be easily seen that $\sigma(C\sigma^{-1}, D) = \sigma(A_X\sigma^{-1}, I) \cup \sigma(I\sigma^{-1}, A_X')$. Hence there's a reciprocal pairing in the spectrum $\sigma(C\sigma^{-1}, D)$.

A relevant question is the connection between the spectrum of $\sigma(C\sigma^{-1}, D)$ and the solutions of (1.1). We show that to (1.4) corresponds a point in $\sigma_P(C\sigma^{-1}, D)$. In fact from (1.4) it follows that a sequence $z \in (\mathbb{R}^{2n})^{\mathbb{Z}}, z(t) \neq 0,\ \lambda_t \in \mathbb{C}$, exists such that $C(t)z(t-1) = D(t)z(t)\lambda_t$ if we define $\lambda^* = \sup\{|\lambda_i|\}$ and a sequence $\{u_n\}$

$$u_1 = \begin{bmatrix} \vdots \\ 0 \\ 0 \\ z(t-1) \\ z(t)\frac{\lambda_{t-1}}{\lambda^*} \\ z(t+1)\frac{\lambda_{t+1}\lambda_t}{\lambda^{*2}} \\ 0 \\ 0 \\ \vdots \end{bmatrix}; \; u_2 = \begin{bmatrix} \vdots \\ 0 \\ z(t-2) \\ z(t-1)\frac{\lambda_{t-1}}{\lambda^*} \\ z(t)\frac{\lambda_{t-1}\lambda_t}{\lambda^{*2}} \\ z(t+1)\frac{\lambda_{t+1}\lambda_t\lambda_{t-1}}{\lambda^{*3}} \\ z(t+2)\frac{\lambda_{t+2}\lambda_{t+1}\lambda_t\lambda_{t-1}}{\lambda^{*4}} \\ 0 \\ \vdots \end{bmatrix}$$

; and so forth, there is a normalized sequence $x_n := u_n/\|u_n\|$ such that

$$\|C\sigma^{-1}x_n - \lambda^* Dx_n\| = \frac{1}{\|x_n\|_{\ell_2}}\left(\|D(t\text{-}n)z(t\text{-}n)\| + \|C(t+n+1)z(t+n)\frac{\lambda_{t+n}\cdots\lambda_{t-n+1}}{\lambda^{*2n}}\|\right)$$

so that it easily follows $\lim_{n\to+\infty}\|C\sigma^{-1}x_n - \lambda^* Dx_n\| = 0$. It follows that $\lambda^*$ is an almost eigenvalues associated to $C\sigma^{-1} - \lambda D$ hence it belongs to $\sigma_P(C\sigma^{-1}, D)$. From (1.4) we conclude that $\dot{+}_{t\in\mathbb{Z}} span \begin{bmatrix} I \\ X(t) \end{bmatrix} \subseteq \dot{+}_{\lambda\in\sigma_P(C\sigma^{-1},D)} E_\lambda(C\sigma^{-1}, D)$.

**Proposition 2.1** *[2] Let $S$ be a Lagrangian subspace of the form $S = \dot{+}_\lambda E_\lambda(C\sigma^{-1}, D)$ with $\lambda \in \sigma_P(C\sigma^{-1}, D)$ where $C,\ D$ are associated to the to the uniformly exponentially stable homogeneous RDE (1.1) (SHRDE) i.e. $Q(t) = 0\forall t$ consider a basis $\begin{bmatrix} X \\ Y \end{bmatrix}$ of $S$: $S = span \begin{bmatrix} X \\ Y \end{bmatrix}$ for which $C(t)\begin{bmatrix} X(t-1) \\ Y(t-1) \end{bmatrix} = D(t)\begin{bmatrix} X(t) \\ Y(t) \end{bmatrix} S(t)$ holds, then $X'Y \le 0$.*

As for the ARE for time-varying RDE two solutions say $X,\ W$ are connected with the eigenspaces relative to matrices $A_X,\ A_W$ (Lemma 3.1 of [5]).

**Proposition 2.2** *[2] Consider two symmetric solutions of (1.1) $X,\ W$, if $|\lambda| \neq 1$ belongs to $\sigma_P(A_X\sigma^{-1}, I)$ and $\lambda \notin \sigma(I\sigma^{-1}, A'_X)$ then $E_\lambda(A_W) \subseteq E_\lambda(A_X)$, $A_X = A_W$ on $E_\lambda(A_W)$, and $E_\lambda(A_W) \subseteq Ker(X - W)$.*

# 3 Parameterization of solutions of the time-varying RDE

Observe that we have the following identity:

$$\begin{aligned} X(t\text{-}1)\text{-}A'_X(t)X(t)A_X(t) = {} & Q(t) + A(t)'X(t)B(t)(R(t) + B(t)'X(t)B(t))^{-1}R(t)\cdot \\ & \cdot(R(t) + B(t)'X(t)B(t))^{-1}B(t)'X(t)A(t) \end{aligned} \tag{3.7}$$

the following condition will be used in the sequel:

*if $|\lambda| = 1$is an almost eigenvalue of $A'$ then the corresponding sequence of almost eigenvectors $\{x_n\}$do not satisfy $\|B'x_n\| \to 0$ as $n \to \infty$.* (3.8)

As explained in [4] this condition is related to uniform reachability of $(A, B)$: a pair $(A, B)$ which satisfies condition (3.8) for all almost eigenvalues of $A$ is said uniformly reachable.

**Remark 3.1** *One could observe that following the rationale of [5] in deriving a parameterization of solutions of RDE means that we are treating infinite-dimensional systems like finite dimensional ones. This is sustained by the fact that from our assumptions the spectrum of $A$ is composed only by eigenvalues i.e. $\sigma_P(A) = \sigma(A)$. In fact, since $A$ is a compact operator,* *it is ensured that $\lambda \neq 0$, $\lambda \in \sigma(A)$ implies $\lambda \in \sigma_P(A)$ and $\lambda = 0$ is always in $\sigma(A)$ [1]. Now if $0 \in \sigma_r(A)$ then $A$ has not dense range, this is a contradiction for $A$ compact on a Hilbert space, then $0 \in \sigma_P(A) \cup \sigma_c(A) \subseteq \sigma_a(A)$. Take a sequence of almost eigenvectors $\{x_n\}$ for $\lambda = 0$ then we can obtain a $u \neq 0$ such that $Au = 0$ hence $0 \in \sigma_P(A)$.*

**Proposition 3.1** *If condition (3.8) holds then we have $E_\lambda(A_X) \subseteq E_\lambda(A) \cap KerQ \cap KerX$ and $A_X = A$ on $E_\lambda(A_X)$.*

**Proof.** In [1] it is proven that a positive integer $n$ exists such that $E_\lambda = Ker(A\sigma^{-1} - \lambda I)^n = Ker(A\sigma^{-1} - \lambda I)^{n+1}$ and $|\lambda| = 1$ and $E_\lambda$ is finite dimensional so $E_\lambda = span\{v_1, \ldots, v_k\}$. Consider the inductive hypothesis $A_X v_s = Av_s$, $Xv_s = 0$, $Qv_s = 0$, $s = 1, \ldots, m$, define $V = span\{v_1, \ldots, v_{m+1}\}$ then $A_X V = VM$;

$$(V^*XV)\sigma^{-1}\text{-}V^*A'_X X A_X V = (V^*XV)\sigma^{-1} - M^*V^*XVM = 0$$

since $\sigma(\bar{M})\sigma(M) = 1$. Then $V^*$(righthand side of (3.7))$V = 0$ hence $v'_{s+1}Qv_{s+1} = 0$ and $B'XAv_{s+1} = 0$ then $A_X v_{s+1} = Av_{s+1}$; $Qv_{s+1} = 0$ moreover $\begin{cases} B'Xv_{s+1} = 0 \\ v'_{s+1}X\sigma^{-1}\text{-}\lambda v'_{s+1}XA = 0 \end{cases}$ from (3.8) follows $v'_{s+1}X = 0$. □

**Proposition 3.2** *If $|\lambda| = 1$ is an almost eigenvalue of $(C\sigma^{-1})' - D'$ and as almost eigenvalue of $A$ satisfies (3.8), then $V_\lambda :=$ (the maximal $A$ invariant subspace $\subseteq E_\lambda(A) \cap KerQ$) $\subseteq KerX$, moreover $E_\lambda(A_X) = V_\lambda$; $A_X = A$ on the set $V_\lambda$.*

**Proof.** From the above proposition we have that $E_\lambda(A_X)$ is A invariant and contained in $E_\lambda(A) \cap KerQ$, hence $E_\lambda(A_X) \subseteq V_\lambda$ we want to show that is equal to $V_\lambda$. There exists a base for which $A = \begin{bmatrix} A_1 & A_{12} \\ 0 & A_2 \end{bmatrix}$ and $Q = \begin{bmatrix} 0 & 0 \\ 0 & Q_2 \end{bmatrix}$, $\sigma A_1 = \{\lambda\}$, observe that if $\lambda$ is an almost eigenvalue of $A$ then the corresponding sequence of eigenvectors $\{x_n\}$ does not verify $\|Q_2 x_n\| \to 0$. We can partition conformingly $C\sigma^{-1}$-$D$:

$$\begin{bmatrix} A_1 & A_2 & 0 & 0 \\ 0 & A_2 & 0 & 0 \\ 0 & Q_2 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \sigma^{-1} - \begin{bmatrix} I & 0 & * & * \\ 0 & I & \tilde{G} & * \\ 0 & 0 & A'_2 & A'_{12} \\ 0 & 0 & 0 & A'_1 \end{bmatrix}.$$

Condition (3.8) implies that $\lambda$ it is not in the spectrum of $\begin{bmatrix} A_2 & 0 \\ Q_2 & I \end{bmatrix} \sigma^{-1} - \begin{bmatrix} I & \tilde{G} \\ 0 & A'_2 \end{bmatrix}$. Observe that $V_\lambda$ is finite dimensional as $\{|\lambda| = 1\} \cap \sigma_P(A)$ is a finite set [1]. So to $V_\lambda$ corresponds an eigenspace of $C\sigma^{-1}$-$D$ formed by the same eigenvalue $\lambda$ of dimension $2dimV_\lambda$, $C\sigma^{-1}$-$D$ can be partitioned in the form (2.6) hence $\dim E_\lambda(A_X) = \dim V_\lambda$ then $E_\lambda(A_X) = V_\lambda$ since $E_\lambda(A_X) \subseteq E_\lambda(A) \cap KerQ \cap KerX$ then $V_\lambda \subseteq KerX$. □

Given $X, W$ symmetric solutions of (1.1) we have the following identity ($\Delta := X - W$):

$$A'_W(t)\Delta(t)A_W(t) - \Delta(t-1) = A'_W(t)\Delta(t)B(t)(I + B'(t)X(t)B(t))^{-1}B'(t)\Delta(t)A_W(t) \quad (3.9)$$

**Proposition 3.3** *Assume that condition (3.8) holds for an indefinite $\lambda$ and $\langle (I+G'XG)x,x\rangle > \mu\|x\|^2\ \forall x\ (\mu>0)$ . If $E_\lambda(A_W)$ is an $A_X$ invariant subspace of $E_\lambda(A_X)$ then $E_\lambda(A_W)\subseteq Ker(X-W)$.*

**Proof.** There is a one to one operator $T$ such that

$$\tilde{A}_W(t)=T(t)^{-1}A_W(t)T(t\text{-}1)=diag(B_1(t),B_2(t)) \tag{3.10}$$

$$\tilde{A}_X(t)=T(t)^{-1}A_X(t)T(t\text{-}1)=\begin{bmatrix} A_1 & A_{12} \\ 0 & A_2 \end{bmatrix} \tag{3.11}$$

with $\sigma(B_1)=\sigma(A_1)=\{\lambda\}$. If $|\lambda|=1$ from proposition 3.2 $E_\lambda(A_X)=V_\lambda\subseteq KerX$, this also holds for $W$ $E_\lambda(A_W)=V_\lambda\subseteq KerW$ so that $E_\lambda(A_W)\subseteq Ker(X-W)$. If $|\lambda|\neq 1$ partition $\Delta$ conformingly to $diag(B_1(t),B_2(t))$: $\Delta=\begin{bmatrix}\Delta_1 & \Delta_{12}\\ \Delta_{12}' & \Delta_2\end{bmatrix}$, from $\Delta\sigma^{-1}=A_X'\Delta A_W$ follows $\Delta_1\sigma^{-1}=A_1'\Delta_1 B_1$ since $1\notin\sigma(A_1)\sigma(\bar{B}_1)$ this Lyapunov equation has a unique solution $\Delta_1=0$ (see [3]). If $y$ is an eigenvector relative to $\sigma(B_1)$ then $y'\Delta y=0$. Consider the eigenspace relative to $\lambda$: $A_W y_i=\lambda y_i+y_{i-1}$ $i=1,\ldots,s$, using an inductive reasoning suppose that $\Delta y_{k-1}=0$, the left hand side of (3.9) gives $y_k'$(righthand side of (3.9)$y_k=0$ since (righthand side of (3.9) $\geq 0$ imply $y_k^* A_W'\Delta G=\lambda^*(\Delta y_k)'G=0$, $(\Delta y_k)'$(lefthand side of (3.9)) $=(\Delta y_k)'(\lambda^* A_W-I)=0$ the hypothesis on the eigenspace of $\lambda$ implies $\Delta y_k=0$. □

**Proposition 3.4** *Given $X,W$ symmetric solutions of (1.1) then:*
1. *$Ker(X-W)$ is invariant for $A_X,\ A_W$.*
2. *$A_X=A_W$ on $Ker(X-W)$.*
3. *If $\Lambda_{in}$ is the set of eigenvalues of $A_W$ relative to $Ker(X-W)$, $\Lambda_{out}$ the remaining ones, then the eigenvalues of $A_X$ are $\Lambda_{in}$, $\Lambda_{out}^{-1}$ respectively.*

**Proof.** Since $\Delta=X-W$ is symmetric from chapter 4 of [1] we obtain that there's a one to one orthogonal operator $S$ such that $S'\Delta S=\tilde{\Delta}=diag\{0,\tilde{\Delta}_2\}$, $\tilde{\Delta}_2$ corresponds to eigenvectors of $\Delta$ relative to eigenvalues $\neq 0$. From $\Delta\sigma^{-1}$-$A_X'\Delta A_W=0$ and $\tilde{A}_X:=S^{-1}A_XS$; $\tilde{A}_W:=S^{-1}A_WS$ follows $\tilde{\Delta}\sigma^{-1}$-$\tilde{A}_X'\tilde{\Delta}\tilde{A}_W=0$; $\tilde{A}_X'=\begin{bmatrix} * & C_{12}\\ * & C_2\end{bmatrix}$; $\tilde{A}_W=\begin{bmatrix} * & * \\ B_{21} & B_2\end{bmatrix}$, then $\tilde{\Delta}_2\sigma^{-1}$-$C_2\tilde{\Delta}_2B_2=0$; $C_{12}\tilde{\Delta}_2B_2=0$; $C_2\tilde{\Delta}_2B_{21}=0$. $\tilde{\Delta}_2$ has no eigenvalues$=0$ then $C_2,B_2$ are non singular, hence $C_{12}=0$, $B_{21}=0$, we obtain $\tilde{A}_X=\begin{bmatrix} A_1 & A_{12}\\ 0 & A_2\end{bmatrix}$ and $\tilde{A}_W=\begin{bmatrix} B_1 & B_{12}\\ 0 & B_2\end{bmatrix}$, hence $Ker\tilde{\Delta}$ invariant for $\tilde{A}_X$, $\tilde{A}_W$ then $Ker(X-W)$ invariant under $A_X,A_W$, this concludes point 1. Point 2 follows from the identity $A_W(t)$-$A_X(t)=(I+\bar{G}(t)X(t))^{-1}\bar{G}(t)\Delta(t)A_W(t)$ and point 1. Finally point 3 follows from 2 and $B_2=\tilde{\Delta}_2^{-1}A_2^{-1}\tilde{\Delta}_2\sigma^{-1}$. □

**Remark 3.2** *With reference to notations of the proof of proposition 3.4 observe that assuming $\langle (I+B'XB)x,x\rangle>\mu\|x\|^2\ \forall x\ (\mu>0)$ and condition (3.8) implies as explained in the proof of proposition 3.3 that $\lambda\notin\sigma(B_2)$, in fact $E_{|\lambda|=1}(A_W)\subseteq Ker(X-W)$ and this implies that $\lambda$ could belong only to $\sigma(B_1)$.*

**Proposition 3.5** *Assume condition (3.8) and* $\langle (I + B'XB)x, x\rangle > \mu\|x\|^2 > 0\ \forall x\ (\mu > 0)$. *Let* $X, W$ *two symmetric solutions of (1.1), define* $E_{\leq}(A_X) := \dot{+}_{|\lambda|\leq 1} E_\lambda(A_X)$; $E_>(A_W) := \dot{+}_{|\lambda|>1} E_\lambda(A_W)$. *The following conditions are equivalent:*
*i.* $X \leq W$
*ii.* $E_{\leq}(A_X) \subseteq Ker(X - W)$
*iii.* $E_>(A_W) \subseteq Ker(X - W)$
*iv.* $Ker(X - W) = E_{\leq}(A_X)\dot{+}E_>(A_W)$.

**Proof.** We make reference to notations of the proof of proposition 3.4. From the previous remark $\lambda \notin \sigma(B_2)$ if $|\lambda| = 1$, moreover $B_2 = \tilde{\Delta}_2^{-1}A_2^{-1}\tilde{\Delta}_2\sigma^{-1}$ implies $\lambda \notin \sigma(A_2)$. Assume $(ii)$, from point 3 of proposition 3.4 we have $|\sigma(A_2)| > 1$, conversely, if $|\sigma(A_2)| > 1$ then $(ii)$ holds. Now assume $(iii)$, from point 3 of proposition 3.4 we have $|\sigma(B_2)| \leq 1$ but remark 3.2 gives $|\sigma(B_2)| < 1$, also the viceversa holds. Since $|\sigma(B_2)| < 1 \Leftrightarrow |\sigma(A_2)| > 1$ because of $B_2 = \tilde{\Delta}_2^{-1}A_2^{-1}\tilde{\Delta}_2\sigma^{-1}$ then $(ii) \Leftrightarrow (iii)$. The following identity is similar to (3.9): $B_2'\tilde{\Delta}_2B_2$-$\tilde{\Delta}_2\sigma^{-1} = B_2'\tilde{\Delta}_2\tilde{G}_2(I + B'XB)^{-1}\tilde{G}_2'\tilde{\Delta}_2B_2 = P \geq 0$ where $\begin{bmatrix}\tilde{G}_1\\ \tilde{G}_2\end{bmatrix} := S^{-1}B$. Assume $(i)$ then $\Delta \leq 0$ iff $\tilde{\Delta}_2 < 0$, if $x \in E_\lambda(B_2)$ then $(|\lambda|^2 - 1)\langle\tilde{\Delta}_2x, x\rangle = \langle Px, x\rangle$ if $\langle Px, x\rangle > 0$ then $|\lambda| < 1$, if $\langle Px, x\rangle = 0$ remember that $1 = |\lambda| \notin \sigma(B_2)$, then $\langle\tilde{\Delta}_2x, x\rangle = 0$ hence $x = 0$, so $|\sigma(B_2)| < 1$. Conversely if $|\sigma(B_2)| < 1$ then (remember $\sigma_a(B_2) = \sigma_P(B_2) = \sigma(B_2)$) proposition 5 of [3] implies $P \geq 0$ so that $\tilde{\Delta}_2 \leq 0$ then $\Delta = X - W \leq 0$. We have obtained $(i) \Leftrightarrow |\sigma(B_2)| < 1 \Leftrightarrow (ii) \Leftrightarrow (iii)$. If we assume $(i)$ and we want to obtain $(iv)$ the reasoning is the same given in [5] while $(iv) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (i)$ is obvious. □
Since $A_X$ is a compact operator of $\mathcal{L}(\ell_2^n, \ell_2^n)$ there are only eigenvalues in the spectrum of $A_X$ ([1]) hence we can decompose $\ell_2^n = E_{\leq}(A_X)\dot{+}E_>(A_X)$. We can define the projections $P_{\leq}(A_X) : \ell_2^n \to E_{\leq}(A_X)$ and $P_>(A_X) : \ell_2^n \to E_>(A_X)$. Now we can give a decomposition for $\ell_2$ symmetric solutions of RDE.

**Theorem 3.1** *Assume* $X, Y, Z$ *symmetric solutions of RDE (1.1), condition (3.8), and* $\langle (I + B'XB)x, x\rangle > \mu\|x\|^2\ \forall x\ (\mu > 0)$. *If* $Y \leq X \leq Z$ *then* $X = ZP_{\leq} + YP_>$.

**Proof.** From $X \leq Z$ we obtain $E_{\leq}(A_X) \subseteq Ker(X - Z)$, from $Y \leq X$ we obtain $E_>(A_X) \subseteq Ker(Y - X)$ then $XP_{\leq} = ZP_{\leq}$ and $XP_> = YP_>$ since $\ell_2^n = E_{\leq}(A_X)\dot{+}E_>(A_X)$ we have $X = XP_{\leq} + XP_> = ZP_{\leq} + YP_>$. □

# References


[1] B. Beauzamy, "Introduction to operator theory and invariant subspaces," North-Holland, Amsterdam, 1988.

[2] N. Carpanese, "A note on the geometry of time-varying Riccati difference equation," *Proc. 40th IEEE Conference on Decision and Control, Orlando, FL (USA)* december 4-7, 2001, 4300-4301.

[3] K.M. Przyluski, “The Lyapunov equation and the problem of stability for linear bounded discrete-time systems in Hilbert space,” *Appl. Math. Optim.* 6, 1980, 97–112.

[4] M. A. Peters and P. A. Iglesias, “A spectral test for observability and reachability of time-varying systems,” *SIAM J. Contr. and Opt.* 37, 1999. 1330–1345.

[5] H. K. Wimmer, Geometry of the discrete-time algebraic Riccati equation, *Journal of Math. Syst., Estimation and Cont.* 2, 1992, 123–132.