OUTPUT REGULATION OF NONLINEAR FUNCTIONAL DIFFERENTIAL EQUATIONS

Emilia M. Fridman
Department of Electrical Engineering/Systems
Tel Aviv University
Ramat Aviv
Tel Aviv, 69978
Israel

Abstract

Output regulation of retarded type nonlinear systems is considered. Regulator equations are derived, which generalize Francis-Byrnes-Isidori equations to the case of systems with delay. It is shown that, under standard assumptions, the regulator problem is solvable if and only if these equations are solvable. In the linear case, the solution of these equations is reduced to linear matrix equations. An example of a delayed Van der Pol equation illustrates the efficiency of the results.

Keywords: time-delay systems, nonlinear systems, output regulation, regulator equations, center manifold

1 Introduction

One of the most important problems in control theory is that of controlling the output of the system so as to achieve asymptotic tracking of prescribed trajectories. This problem of output regulation has been studied by many authors (see e.g. a survey paper by Byrnes and Isidori [2] and the references therein). In the linear case, Francis [5] showed that the solvability of a multivariable regulator problem corresponds to the solvability of a system of two linear matrix equations. In the nonlinear case, Isidori and Byrnes [10] proved that the solvability of the output regulation problem is equivalent to the solvability of a set of partial differential and algebraic equations. This set of partial differential and algebraic equations is now known as the *regulator equations* or *Francis-Isidori-Byrnes equations*. For linear infinite-dimensional control systems a solution of the regulator problem was introduced in [3]. This solution was given in terms of the operator regulator equations.

In the present paper, we consider output regulation of *nonlinear* systems with delay. These are infinite-dimensional systems, which are important in applications. We generalize the result of [10] to time-delay systems by showing that the problem is solvable iff certain regulator equations are solvable. These equations consist of partial differential equations for a center manifold of the closed-loop system with delay and of an algebraic equation. In the linear case the solution of these equations is reduced to linear matrix equations. We find the relation between our linear equations and the operator regulator equations of [3] and we analyze the solvability of the linear equations. An example of delayed Van der Pol equation illustrates the developed theory.

Notations. Let R^m be Euclidean space with the norm $|\cdot|$ and $C^m[a,b]$ be the space of continuous functions $\phi:[a,b]\rightarrow R^m$ with the supremum norm $||\cdot||$.

Denote by $x_t(\theta)=x(t+\theta)$ $(\theta\in[-h;0])$.

$L_2([-h,0],R^n)$ is the space of square integrable R^n valued functions with the corresponding norm.

$W^{1,2}([-h,0],R^n)$ is the space of absolutely continuous R^n valued functions on $[-h,0]$ with square integrable derivatives.

The transpose of a matrix M is written M'.

2 Problem Formulation

We consider a nonlinear system modeled by equations of the form

$$\dot{x}(t)=f(x_t,u(t),w(t)),\quad x(\theta)=\phi(\theta),\,\theta\in[-h,0] \tag{2.1}$$

$$e(t)=h(x_t,w(t)) \tag{2.2}$$

with state $x(t)\in R^n$, initial function $\phi\in C^n[-h,0]$, control input $u(t)\in R^m$, exogenous input $w(t)\in R^r$ and tracking error $e(t)\in R^p$. The exogenous input is generated by an autonomous dynamical system of the form

$$\dot{w}(t)=s(w(t)) \tag{2.3}$$

The functions $f:V\rightarrow R^n$, $s:W\rightarrow R^r$, $h:Y\rightarrow R^p$ are smooth (i.e. C^∞) mappings, where $V\subset C^n[-h,0]\times R^m\times R^r,\ W\subset R^r, Y\subset C^n[-h,0]\times R^r$ are some neighborhoods of the origin of the corresponding spaces. We assume that $f(0,0,0)=0$, $s(0)=0$, $h(0,0)=0$. Thus, for $u=0$, the system (2.1) has an equilibrium state $(x,w)=(0,0)$ with zero error (2.2).

We consider both, a state-feedback and an error-feedback regulator problems.

Problem 1 (State-Feedback Regulator Problem): Find a state-feedback control law

$$u(t)=\alpha(x_t,w(t)), \tag{2.4}$$

where $\alpha(x_t,w(t)):Y\rightarrow R^m$ is a $C^k(k\geq 2)$ function and $\alpha(0,0)=0$ such that :

1a) the equilibrium $x(t)\equiv 0$ of

$$\dot{x}(t)=f(x_t,\alpha(x_t,0),0),$$

is exponentially stable;

1b) there exists a neighborhood $Y \subset C^n[-h,0] \times W$ of the origin such that, the solution of the closed-loop system

$$\dot{x}(t) = f(x_t, \alpha(x_t, w(t)), w(t)), \quad \dot{w}(t) = s(w(t)) \tag{2.5}$$

satisfies

$$\lim_{t\to\infty} h(x_t, w(t)) = 0. \tag{2.6}$$

Problem 2 (Error-Feedback Regulator Problem): Find an error-feedback controller

$$u = \Theta(z_t), \quad \dot{z}(t) = \eta(z_t, e(t)), \tag{2.7}$$

where $z(t) \in R^\nu$ and where η and Θ are $C^k (k \geq 2)$ mappings, such that:

2a) the equilibrium $(x(t), z(t)) \equiv 0$ of

$$\dot{x}(t) = f(x_t, \Theta(z_t), 0), \quad \dot{z}(t) = \eta(z_t, h(x_t, 0))$$

is exponentially stable;

2b) there exists a neighborhood $Z \subset C^n[-h,0] \times C^\nu[-h,0] \times W$ of the origin such that, the solution of the closed-loop system

$$\dot{x}(t) = f(x_t, \Theta(z_t), w(t)), \quad \dot{z}(t) = \eta(z_t, h(x_t, w(t))), \quad \dot{w}(t) = s(w(t)) \tag{2.8}$$

satisfies (2.6).

3 Linearized Problem and Assumptions

Smooth functions f, h, α, Θ and η can be represented in the form

$$\begin{aligned}
&f(x_0, u, w) = \mathrm{A}x_0 + Bu + Pw + O(x_0, u, w)^2, \\
&h(x_0, w) = \mathrm{C}x_0 + Qw + O(x_0, w)^2, \\
&\alpha(x_0, w) = \mathrm{K}x_0 + Lw(t) + O(x_0, w)^2, \\
&\Theta(z_0) = \mathrm{H}z_0 + O(z_0)^2, \quad \eta(z_0, e) = \mathrm{F}z_0 + Ge + O(z_0, e)^2,
\end{aligned}$$

where $O(\cdot)^2$ vanishes at the origin with its first-order derivatives. The linear mappings $\mathrm{A} : C^n[-h,0] \to R^n$ and $\mathrm{C} : C^n[-h,0] \to R^p$ by Riesz theorem can be represented in the form of Stieltjes integrals [9]:

$$\mathrm{A}\phi = \int_{-h}^{0} d[\mu(\theta)]\phi(\theta), \quad \mathrm{C}\phi = \int_{-h}^{0} d[\zeta(\theta)]\phi(\theta), \tag{3.9}$$

with $n \times n$ and $p \times n$-matrix functions μ and ζ of bounded variations. A similar representation can be written for the linear mappings $\mathrm{K} : C^n[-h,0] \to R^m$, $\mathrm{H} : C^n[-h,0] \to R^m$ and $\mathrm{F} : C^n[-h,0] \to R^n$.

The linearized system is given by

$$\begin{aligned} \dot{x}(t) &= \mathrm{A}x_t + Bu(t) + Pw(t), \\ \dot{w}(t) &= Sw(t), \\ e(t) &= \mathrm{C}x_t + Qw(t). \end{aligned} \tag{3.10}$$

The linearized state-feedback and error-feedback controllers have the form

$$u(t) = \mathrm{K}x_t + Lw(t) \tag{3.11}$$

and

$$u(t) = \mathrm{H}z_t, \quad \dot{z}(t) = \mathrm{F}z_t + Ge(t). \tag{3.12}$$

respectively.

Similarly to the case without delay [10] we assume the following:

H1. The exosystem (2.3) is neutrally stable (i.e. Lyapunov stable in forward and backward time, and thus S has all its eigenvalues on the imaginary axis).

H2. The pair $\{\mathrm{A}, B\}$ is stabilizable, i.e. there exists a linear bounded mapping $\mathrm{K} : C^n[-h, 0] \to R^m$ such that the system

$$\dot{x}(t) = (\mathrm{A} + B\mathrm{K})x_t \tag{3.13}$$

is asymptotically stable.

H3. The pair

$$\begin{bmatrix} \mathrm{A} & P \\ 0 & S \end{bmatrix}, \qquad [\mathrm{C} \quad Q]$$

is detectable, i. e. there exists a $(n + r) \times p$-matrix G such that the system

$$\begin{bmatrix} \dot{\xi}_1(t) \\ \dot{\xi}_2(t) \end{bmatrix} = \left\{ \begin{bmatrix} \mathrm{A} & P \\ 0 & S \end{bmatrix} + G[\mathrm{C} \quad Q] \right\} \begin{bmatrix} \xi_{1t} \\ \xi_2(t) \end{bmatrix}, \tag{3.14}$$

where $\xi_1(t) \in R^n,\ \xi_2(t) \in R^r$, is asymptotically stable.

4 Solution of the Regulator Problems

4.1 Center manifold of the closed-loop system.

The solution of the output regulation problem is usually based on the application of the center manifold theory. The existence, smoothness and the attractiveness of the center manifold for systems with delay were proved e.g. in [8], [9]. A partial differential equation for the function, determining the center manifold for system with delay was derived in [12], [6], [1]. Modifying these results to the closed-loop system (2.5) we obtain the following:

Lemma 4.1. *Assume that all eigenvalues of S are on the imaginary axis and that for some $\alpha(x_t, w)$ condition 1a) holds. Then*
(i) *the closed-loop system (2.5) has a local invariant (center) manifold $x_t(\theta) = \pi(w(t))(\theta)$, where $\pi : W_0 \to C^n[-h, 0]$ $(0 \in W_0 \subset W \subset R^r)$ is a C^k mapping with $\pi(0)(\theta) \equiv 0$;*
(ii) *the center manifold is locally attractive, i.e. satisfies*

$$||x_t - \pi(w(t))|| \leq Me^{-at}||x_0 - \pi(w(t))||, \quad M > 0,\ a > 0 \tag{4.15}$$

for all $x_0, w(0)$ sufficiently close to 0 and all $t \geq 0$.
(iii) *A C^1 mapping $\pi : W_0 \to C^n[-h, 0],\ \pi(0) = 0$ defines a center manifold $x_t = \pi(w(t))$ of (2.5) if and only if it satisfies the following system of partial differential equations $\forall w \in W_0$*

$$\frac{\partial \pi(w)(\theta)}{\partial w} s(w) = \frac{\partial \pi(w)(\theta)}{\partial \theta}, \quad \theta \in [-h, 0], \tag{4.16}$$

$$\frac{\partial \pi(w)(0)}{\partial w} s(w) = f(\pi(w), \alpha(\pi(w), w), w). \tag{4.17}$$

Proof. (i), (ii): The closed-loop system (2.5) has the form

$$\begin{aligned} \dot{x}(t) &= (\mathrm{A} + B\mathrm{K})x_t + (P + BL)w(t) + O(x_t, w(t))^2, \\ \dot{w}(t) &= Sw(t) + O(w(t))^2. \end{aligned} \tag{4.18}$$

By assumption, the zeros of the characteristic equation corresponding to (3.13) are in C^-, and the eigenvalues of the matrix S are on the imaginary axis. Since we are interested only in the behavior of (4.18) in a small neighborhood of origin, we may suppose, without loss of generality, that for some $\rho > 0$

$$O(x_0, w)^2 = 0,\ O(w)^2 = 0 \text{ for } |w| \geq \rho,\ x_0 \in C^n[-h, 0].$$

From the theory of invariant manifolds (see e.g. [8] or chapter 10.2 in [9]) applied to these new functions $O(x_0, w)^2$ and $O(w)^2$ it follows that the system (4.18) has a local attractive center manifold $x_t = \pi(w(t))$, where $\pi : W_0 \to C^n[-h, 0]$ is a C^k mapping vanishing in zero. The flow on this manifold is governed by the second equation of (4.18). Let $X(t),\ t \in [-h, \infty)$ be a fundamental matrix of (3.13) and $w(t)$ be a solution of the second equation of (4.18) with the initial condition $w(0) = w_0$. The theory of center manifolds implies that

$$\pi(w_0)(\theta) = \int_{-\infty}^{0} X(-s + \theta)[(P + BL)w(s) + O(\pi(w(s)), w(s))^2]ds, \tag{4.19}$$

and that (4.19) has a unique smooth solution π for small enough ρ.

(iii) System (2.5) with C^1 initial function $x_0 = \phi$ that satisfies $\dot{\phi}(0) = f(\phi, \alpha(\phi, w_0), w_0)), w_0)$ is equivalent to

$$\begin{aligned} \tfrac{\partial x_t(\theta)}{\partial t} &= \tfrac{\partial x_t(\theta)}{\partial \theta}, \quad x_0 = \phi, \quad \theta \in [-h, 0], \quad t \geq 0, \\ \tfrac{\partial x_t(0)}{\partial t} &= f(x_t, \alpha(x_t, w(t)), w(t)), \quad \dot{w}(t) = s(w(t)). \end{aligned} \tag{4.20}$$

For a C^1 mapping π and for $w(t)$, satisfying (2.3), we find that for each $\theta \in [-h, 0]$

$$\frac{d}{dt}[\pi(w(t))(\theta)] = \frac{\partial \pi(wi(t))(\theta)}{\partial w} s(w(t)). \tag{4.21}$$

Let a C^1 mapping π determines a center manifold of (4.18). Then $x_t = \pi(w(t))$ satisfies (2.5) and, hence (4.20). Substituting $x_t = \pi(w(t)),\ w(0) = w,\ t \geq 0$ into (4.20) and setting further $t = 0^+$, we obtain that for all $w \in W_0$, $\pi(w)(\theta)$ is differentiable in $\theta \in [-h, 0]$ and π satisfies (4.16) and (4.17).

Conversely, let π satisfies (4.16) and (4.17). Substitute $w = w(t)$ into (4.16), (4.17), where $w(t)$ is a solution of (2.3), then $x_t = \pi(w(t))$ satisfies (4.20) (and thus (2.5)) and therefore π determines the invariant manifold of (2.5). $\square$

Remark 4.1. *Approximate solution to (4.16), (4.17) can be found in a form of series expansions in the powers of w (similarly to [8], [12], [1]).*

4.2 State-Feedback Regulator Problem

Applying Lemma 1, we obtain regulator equations by using arguments of [10].

Lemma 4.2. *Under H1 assume that for some $\alpha(x_t, w)$ condition 1a) holds. Then, condition 1b) is also fulfilled iff there exists a $C^k (k \geq 2)$ mapping $\pi : W_0 \to C^n[-h, 0],\ \pi(0) = 0$ satisfying (4.16), (4.17) and the algebraic equation*

$$h(\pi(w), w) = 0. \tag{4.22}$$

Proof. The proof is similar to Lemma 1 of [10] and it is based on Lemma 1 above. The closed-loop system (2.5) has a center manifold. By H1 no trajectory on this manifold converges to zero. Then 1b) holds only if this manifold is annihilated by the error map e, i.e. only if (4.22) holds. On the other hand, since the center manifold is locally attractive, (4.22) guarantees that 1b) is satisfied. $\square$

Theorem 4.1. *Under H1 and H2, the state-feedback regulator problem is solvable if and only if there exist $C^k (k \geq 2)$ mappings $x_0(\theta) = \pi(w)(\theta)$, with $\pi(0)(\theta) = 0$, and $u = c(w)$, with $c(0) = 0$, both defined in a neighborhood $W \subset R^r$ of the origin, satisfying the conditions*

$$\frac{\partial \pi(w)(\theta)}{\partial w} s(w) = \frac{\partial \pi(w)(\theta)}{\partial \theta},\ \ \theta \in [-h, 0], \tag{4.23}$$

$$\frac{\partial \pi(w)(0)}{\partial w} s(w) = f(\pi(w), c(w), w), \tag{4.24}$$

$$h(\pi(w), w) = 0. \tag{4.25}$$

Suppose that π and c satisfy (4.23)-(4.25), then the state-feedback

$$u = \alpha(x_t, w(t)) = c(w(t)) + K[x_t - \pi(w(t))], \tag{4.26}$$

where K is a stabilizing gain which is defined in H2, solves the state-feedback regulator problem.

Proof. The necessity follows immediately from Lemma 2. For the sufficiency consider the state-feedback (4.26). This choice satisfies 1a), since

$$f(x_t, \alpha(x_t, 0), 0) = (\mathrm{A} + B\mathrm{K})x_t + O(x_t)^2.$$

Moreover, by construction

$$\alpha(\pi(w), w) = c(w)$$

and therefore, (4.23), (4.24) reduces to (4.16), (4.17). From (4.25) by Lemma 2 it follows that condition 1b) is also fulfilled. $\square$

4.3 Error-Feedback Regulator Problem

Applying Lemma 1 to the system (2.8), we obtain the following:

Lemma 4.3. *Assume that all eigenvalues of S are on the imaginary axis and that for some $\theta(z_t)$ and $\eta(z_t, e)$ condition 2a) holds. Then*
(i) *the closed-loop system (2.8) has a local invariant (center) manifold $x_t(\theta) = \pi(w(t))(\theta)$, $z_t(\theta) = \sigma(w(t))(\theta)$, where $\pi : W_0 \to C^n[-h, 0]$, $\sigma : W_0 \to C^\nu[-h, 0]$ $(0 \in W_0 \subset W \subset R^r)$ are C^k mappings with $\pi(0)(\theta) \equiv 0$, $\sigma(0)(\theta) \equiv 0$;*
(ii) *the center manifold is locally attractive, i.e. satisfies*

$$||x_t - \pi(w(t))|| + ||z_t - \sigma(w(t))|| \le Me^{-at}(||x_0 - \pi(w(t))|| + ||z_0 - \sigma(w(t))||), \quad M > 0,\ a > 0 \tag{4.27}$$

for all $x_0, z_0, w(0)$ sufficiently close to 0 and all $t \ge 0$.
(iii) *C^1 mappings $\pi : W_0 \to C^n[-h, 0]$, $\pi(0) = 0$, $\sigma : W_0 \to C^n[-h, 0]$, $\sigma(0) = 0$ define center manifold $x_t = \pi(w(t))$, $z_t = \sigma(w(t))$ of (2.8) if and only if it satisfies the following system of partial differential equations $\forall w \in W_0$*

$$\frac{\partial \pi(w)(\theta)}{\partial w} s(w) = \frac{\partial \pi(w)(\theta)}{\partial \theta}, \quad \frac{\partial \sigma(w)(\theta)}{\partial w} s(w) = \frac{\partial \sigma(w)(\theta)}{\partial \theta}, \quad \theta \in [-h, 0] \tag{4.28}$$

$$\frac{\partial \pi(w)(0)}{\partial w} s(w) = f(\pi(w), \theta(\sigma(w)), w), \quad \frac{\partial \sigma(w)(0)}{\partial w} s(w) = \eta(\sigma(w), 0). \tag{4.29}$$

Similarly to Lemma 2, the following lemma can be proved

Lemma 4.4. *Under H1, assume that for some $\Theta(z_t)$ and $\eta(z_t, e)$ condition 2a) holds. Then, condition 2b) is also fulfilled iff there exist a $C^k (k \ge 2)$ mapping $\pi : W_0 \to C^n[-h, 0]$, $\pi(0) = 0$, $\sigma : W_0 \to C^\nu[-h, 0]$, $\sigma(0) = 0$ satisfying (4.28), (4.29) and the algebraic equation (4.22).*

From the latter lemmas we deduce a necessary and sufficient condition for the solvability of the error-feedback regulator problem

Theorem 4.2. *Under H1, H2 and H3, the error-feedback regulator problem is solvable if and only if there exist $C^k (k \ge 2)$ mappings $x_0(\theta) = \pi(w)(\theta)$, with $\pi(0)(\theta) = 0$, and $u = c(w)$,* *with $c(0)=0$, both defined in a neighborhood $W \subset R^r$ of the origin, satisfying the conditions (4.23)-(4.25). Suppose that π and c satisfy (4.23)-(4.25), and that a linear bounded mapping $H : C^n[-h,0] \to R^m$ is such that the system*

$$\dot{x}(t) = (A + BH)x_t \tag{4.30}$$

is asymptotically stable. Then the error-feedback (2.7), where

$$\begin{aligned} &z = col\{z_1, z_2\},\ \ \eta = col\{\eta_1, \eta_2\}, \\ &u = \Theta(z_t) = c(z_2(t)) + H[z_{1t} - \pi(z_2(t))], \\ &\eta_1(z_{1t}, z_2(t), e(t)) = f(z_{1t}, \Theta(z_t), z_2(t)) - G_1(h(z_{1t}, z_2(t)) - e(t)), \\ &\eta_2(z_{1t}, z_2(t), e(t)) = s(z_2(t)) - G_2(h(z_{1t}, z_2(t)) - e(t)), \end{aligned} \tag{4.31}$$

and where $G = col\{G_1, G_2\}$ is defined in H3, solves the regulator problem.

Proof. The necessity follows immediately from Lemma 4. For the sufficiency we note, that there exist a linear bounded functional H and a matrix $G = col\{G_1, G_2\}$ such that (4.30) and (3.14) are asymptotically stable. A standard calculation shows that for any $m \times r$-matrix K, the characteristic quasipolynomial that corresponds to the system

$$\begin{bmatrix} \dot{x}(t) \\ \dot{z}_1(t) \\ \dot{z}_2(t) \end{bmatrix} = \begin{bmatrix} \mathrm{A} & B\mathrm{H} & BK \\ G_1\mathrm{C} & \mathrm{A} + B\mathrm{H} - G_1\mathrm{C} & P + BK - G_1Q \\ G_2\mathrm{C} & -G_2\mathrm{C} & S - G_2Q \end{bmatrix} \begin{bmatrix} x_t \\ z_{1t} \\ z_2(t) \end{bmatrix} \tag{4.32}$$

is equal to the product of the characteristic quasipolynomials that correspond to (4.30) and (3.14) respectively. Therefore, (4.32) is asymptotically stable.

Consider the error-feedback controller of (2.7), (4.31). The linearized system corresponding to the closed-loop system (2.8) has exactly the form of (4.32), where

$$K = \left[\frac{\partial c}{\partial w}\right]_{w=0} - \mathrm{H}\left[\frac{\partial \pi}{\partial w}\right]_{w=0}.$$

Thus requirement 2a) is satisfied. By construction (4.23)-(4.24) imply (4.28)-(4.29) with $\sigma(w) = col\{\pi(w), w\}$. Thus requirement 2b) follows from Lemma 4. $\Box$

5 Linear Case.

5.1 Linear Regulator equations.

Consider the linear regulator problem (3.10). In the linear case the invariant manifold has a form $x_t = \Pi(\theta)w(t)$, where Π is an $n \times r$ matrix function continuously differentiable in $\theta \in [-h, 0]$. From Theorems 1 and 2 it follows, that the linear problem (3.10) is solvable iff there exists Π and an $m \times r$-matrix Γ that satisfy the following system

$$\frac{\partial \Pi(\theta)}{\partial \theta} = \Pi(\theta)S, \quad \theta \in [-h, 0], \tag{5.33}$$

$$\Pi(0)S = \int_{-h}^{0} d[\mu(\theta)]\Pi(\theta) + B\Gamma + P, \tag{5.34}$$

$$\int_{-h}^{0} d[\zeta(\theta)]\Pi(\theta) + Q = 0. \tag{5.35}$$

Eq. (5.33) yields

$$\Pi(\theta) = \Pi(0)e^{S\theta}. \tag{5.36}$$

Substituting (5.36) into (5.35), we obtain the following linear algebraic system for initial value $\Pi(0)$:

$$\begin{array}{l} \Pi(0)S = \int_{-h}^{0} d[\mu(\theta)]\Pi(0)e^{S\theta} + B\Gamma + P, \\ \int_{-h}^{0} d[\zeta(\theta)]\Pi(0)e^{S\theta} + Q = 0. \end{array} \tag{5.37}$$

The latter system is a generalization of Francis equations [5] to the case of retarded systems.

We consider now a particular, but important in applications case of (3.10) with

$$\mathrm{A}x_t = \sum_{i=0}^{k} A_i x(t-h_i) + \int_{-h}^{0} A_d(\theta)x(t+\theta)d\theta, \quad \mathrm{C}x_t = \sum_{i=0}^{k} C_i x(t-h_i) + \int_{-h}^{0} C_d(\theta)x(t+\theta)d\theta, \tag{5.38}$$

where $0 = h_0 < h_1 < \ldots < h_k \leq h$, A_d and C_d are piecewise continuous matrix functions and where A_i and C_i are constant matrices of the appropriate dimensions. In this case (5.37) has the form:

$$\begin{array}{l} \Pi(0)S = \sum_{i=0}^{k} A_i\Pi(0)e^{-Sh_i} + \int_{-h}^{0} A_d(\theta)\Pi(0)e^{S\theta}d\theta + B\Gamma + P, \\ \sum_{i=0}^{k} C_i\Pi(0)e^{-Sh_i} + \int_{-h}^{0} C_d(\theta)\Pi(0)e^{S\theta}d\theta + Q = 0. \end{array} \tag{5.39}$$

Corollary 5.1. *Under H1 and H2, the linear state-feedback regulator problem (3.10) ((3.10) and (5.38)) is solvable if and only if there exist $n \times r$ and $m \times r$-matrices $\Pi(0)$ and Γ which solve the linear matrix equations (5.37) ((5.39)).*

For the case of error-feedback regulator problem, the similar result holds under H1, H2 and H3.

5.2 Relation to the Operator Regulator Equations.

We consider (5.38), where there is no discrete delay in the equation for the error e, i.e.

$$\mathrm{C}x_t = C_0 x(t) + \int_{-h}^{0} C_d(s)x(t+s)ds. \tag{5.40}$$

We show that in this case the linear problem may be formulated in the form of an infinite dimensional system, defined on a Hilbert space with the bounded output operator, and regulator equations (5.39) follow from the operator regulator equations obtained in [3]. Eq.

(3.10) may be represented in the form of an evolution equation (see e.g. [4]) by introducing a Hilbert space

$$M_2 = R^n \times L_2([-h, 0]; R^n)$$

endowed with the inner product

$$< \phi, \psi >= \phi^{0'} \psi^0 + \int_{-h}^{0} \phi^{1'}(\theta)\psi(\theta)d\theta, \ \phi = (\phi^0, \phi^1) \in M^2, \ \psi = (\psi^0, \psi^1) \in M^2.$$

The infinitesimal generator corresponding to the homogeneous system $\dot{x}(t) = \mathrm{A}x_t$ is characterized by

$$\mathcal{A}(\phi^0, \phi^1) = (\mathrm{A}\phi^1, \dot{\phi}^1), \ (\phi^0, \phi^1) \in D(\mathcal{A}),$$
$$D(\mathcal{A}) = \{(\phi^0, \phi^1) \in M^2 : \phi^0 = \phi^1(0), \ \phi^1 \in W^{1,2}(-h, 0; R^n)\}.$$

Note that in the case of nonzero C_i for some $i > 0$, the linear operator $\mathrm{C} : M^2 \to R^n$ is unbounded, while (5.40) is bounded. Eq. (3.10), (5.38), (5.40) can be written in the form of the evolution equation

$$\begin{aligned} &\dot{\bar{x}}(t) = \mathcal{A}\bar{x}(t) + (Bu(t), 0) + (Pw(t), 0), \\ &\dot{w}(t) = Sw(t), \quad \mathrm{C}x_t + Qw(t) = 0. \end{aligned} \tag{5.41}$$

In [3] the following regulator equations were derived in the case of bounded input and bounded output operators:

$$\Pi S = \mathcal{A}\Pi + (B, 0)\Gamma + (P, 0), \quad \mathrm{C}\Pi + Q = 0, \tag{5.42}$$

where $\Pi : R^r \to M^2$ is a linear bounded operator, Γ is an $m \times r$-matrix. It is clear that (5.42) is equivalent to (5.33)-(5.35).

5.3 On the solvability of the linear regulator equations.

As in [3], we consider the case of $p = m$. Solvability of the regulator equations of infinite dimensional linear system with bounded input and output operators was studied in [3]. We apply results of [3] to systems with delay. Consider the transfer function

$$G(s) = (C_0 + \int_{-h}^{0} C_d(\theta)e^{s\theta}d\theta)(sI - \int_{-h}^{0} d[\mu(\theta)]e^{s\theta})^{-1}B, \tag{5.43}$$

which corresponds to the linear system (3.10), (5.40) with $P = 0$ and $Q = 0$. A transmission zero of this linear system is such $\lambda \in C$ that $detG(\lambda) = 0$. From [3] it follows

Proposition 5.1. *Under H1 and H2 the output regulation via bounded state-feedback of (3.10), where C is given by (5.40), is achievable and thus the regulator equations (5.37) are solvable if and only if* $detG(\lambda) \neq 0$ *for all eigenvalue* λ *of* S.

In the case of (5.38) with unbounded input and output we assume that the regulator problem for (3.10) without delay, i.e. with $h = 0$ is solvable. This is equivalent to the following assumption

A1. $detG_0(\lambda) \neq 0$ for all eigenvalues λ of S, where

$$G_0(\lambda) = (\sum_{i=0}^{k} C_i)(\lambda I - \sum_{i=0}^{k} A_i)^{-1} B.$$

Under A1 the linear regulator equations

$$\Pi_0 S = (\sum_{i=0}^{k} A_i)\Pi_0 + B\Gamma + P, \quad (\sum_{i=0}^{k} C_i)\Pi_0 + Q = 0,$$

where Π_0 and Γ are constant matrices, that correspond to the non-delay system

$$\dot{x}(t) = (\sum_{i=0}^{k} A_i)x(t) + Bu(t), \quad e = (\sum_{i=0}^{k} C_i)x(t) + Qw(t)$$

are solvable. Then, by the implicit function theorem for all small enough $h > 0$ (5.39) is solvable. We have:

Proposition 5.2. *Assume H1, H2 and A1. The regulator equations (5.39) are solvable and the output regulation via state-feedback of (3.10) is achievable for all small enough h.*

6 Example

Consider the forced delayed Van der Pol Equation

$$\begin{aligned} &\dot{x}_1(t) = -x_2(t-h), \\ &\dot{x}_2(t) = x_1(t-h) + ax_2(t-h) + bx_2^3(t-h) + u(t), \\ &e(t) = x_1 - w_1, \end{aligned} \tag{6.44}$$

with the exosystem

$$\begin{bmatrix} \dot{w}_1 \\ \dot{w}_2 \end{bmatrix} = \begin{bmatrix} 0 & \Omega \\ -\Omega & 0 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}, \ \Omega \in [0, 2\pi]. \tag{6.45}$$

The unforced equation (6.44) was studied by Murakami [11]. It was shown that for $a > 0$, $b < 0$, while the system without delay has a stable limit cycle, delayed Van der Pol Equation may exhibit a chaotic behavior. In the case of $a < 0$, $b < 0$, the equation without delay is asymptotically stable, whereas for $h > 0$ there may appear a periodic solution. Output regulation of (6.44), (6.45) without delay was considered in [2].

Regulator equations for (6.44), (6.45) with $w = col\{w_1, w_2\}$, $\pi = col\{\pi_1, \pi_2\}$ have the form:

$$\frac{\partial \pi(w)(\theta)}{\partial w}\begin{bmatrix} 0 & \Omega \\ -\Omega & 0 \end{bmatrix} w = \frac{\partial \pi(w)(\theta)}{\partial \theta}, \ \theta \in [-h, 0], \tag{6.46}$$

$$\frac{\partial \pi(w)(0)}{\partial w}\begin{bmatrix} 0 & \Omega \\ -\Omega & 0 \end{bmatrix} w = \begin{bmatrix} -\pi_2(w)(-h) \\ \pi_1(w)(-h) + a\pi_2(w)(-h) + b\pi_2^3(w)(-h) + c(w) \end{bmatrix}, \tag{6.47}$$

$$\pi_1(w)(0) = w_1. \tag{6.48}$$

The linearized problem (6.44), (6.45) (where $b = 0$) is solvable by Proposition 1 since $G(s) = e^{-sh} \neq 0$ for $s = \pm\Omega j$. Substituting (6.48) into the first row of (6.47) we find

$$\pi_2(w)(-h) = -\Omega w_2. \tag{6.49}$$

Solving the boundary value problem (6.46), (6.48) and (6.49) we obtain

$$\pi(w)(\theta) = \begin{bmatrix} \cos\Omega\theta & \sin\Omega\theta \\ \Omega\sin(\Omega(h+\theta)) & -\Omega\cos(\Omega(h+\theta)) \end{bmatrix} w. \tag{6.50}$$

Finally from the second row of (6.47) and from (6.50) we derive

$$c(w) = (\Omega^2 - 1)\cos\Omega h \cdot w_1 + [(\Omega^2 \ + 1)\sin\Omega h + a\Omega]w_2 + b\Omega^3 w_2^3. \tag{6.51}$$

For $h = 0$ the controller $u(t) = -(3 + a)x_2(t)$ stabilisizes the linearized system (6.44). Then for all small enough $h > 0$ this controller is stabilizing for the linearized system (6.44) and thus the corresponding state-feedback may be chosen as follows:

$$u = c(w) - (3 + a)[x_2(t) - \Omega(sin\Omega h \cdot w_1 - \cos\Omega h \cdot w_2)]. \tag{6.52}$$

We made numerical simulations of (6.44), (6.52) for $a = 1$, $b = -1$, $\Omega = 0.5$, $w_1 = cos\Omega t$, $h = 1$ and $x_0 = 0$. Note that by the stability condition of [7], this state-feedback stabilizes the linearized system. Plots of the output $x_1(t)$ and of the reference signal $w_1(t)$ are given in Figure 1. It is clear that $x_1(t)$ asymptotically approaches $w_1(t)$.

7 Conclusions

The geometric theory of output regulation is generalized to nonlinear systems with delay. It is shown that the state-feedback and the error-feedback regulator problems are solvable, under the standard assumptions on stabilizability and detectability of the linearized system, if and only if a set of regulator equations is solvable. This set consists of partial differential and algebraic equations. In the linear case these equations are reduced to the linear matrix equations. The solvability of the linear equations is analyzed in terms of the transmission zeros of the system.

The issues of the solvability of the nonlinear regulator equations and of approximate solutions to these equations are currently under study.

Acknowledgement: Research is supported by the Ministry of Absorbtion of Israel.

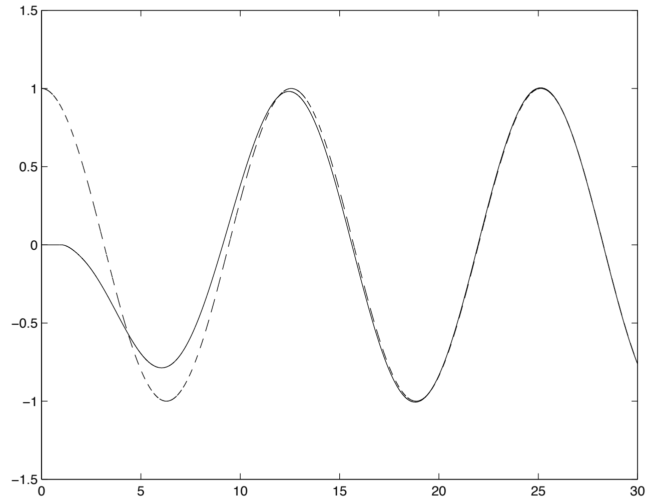

Figure 1: $x_1(t)$ – solid line, $w_1(t)$ – dashed line

References

[1] M. Ait Babram, O. Arino and M. Hbid,“ Computational scheme of a center manifold for neutral functional differential equations, ” *J. Math. Anal. Appl.* 258, 2001, 396–414.

[2] C.I.Byrnes and A. Isidori, “ Output regulation for nonlinear systems: an overview,” *Int. J. of Rob. and Nonlin. Cont.* 10, 2000, 323–337.

[3] C.I.Byrnes, I. G. Lauko, D. S. Gilliam and V. I. Shubov, “Output Regulation for Linear Distributed parameter systems,” *IEEE Trans. On Aut. Cont.*, 45 , 2000 ,2236–2252.

[4] M. Delfour, McCalla C. and Mitter S., “Stability and the infinite-time quadratic cost problem for linear hereditary differential systems,” *SIAM J. Control*, 13, 1975 ,48–88.

[5] B.A. Francis, “ The Linear Multivariable Regulator Problem,” *SIAM J. on Cont. and Optim.*, 15, 1977, 486–505.

[6] E. Fridman, “Asymptotics of integral manifolds and decomposition of singularly perturbed systems of neutral type,” *Differential equations* 26, 1990, 457–467.

[7] E. Fridman and U. Shaked, “ A descriptor system approach to H_∞ control of time-delay systems,” *IEEE Trans. on Automat. Contr.*, 47, 2002.

[8] J. Hale, “Critical cases for neutral functional differential equations,” *J. Differential Eqns.* 10, 1971, 59-82.

[9] J. Hale and S. Verduyn Lunel, “ Introduction to Functional Differential Equations, ” Springer-Verlag - New York, 1993.

[10] A. Isidori and C.I. Byrnes, " Output regulation of nonlinear systems, " *IEEE Trans. On Autom. Control* 35, 1990, 131-140.

[11] K. Murakami, "Bifurcated periodic solutions for delayed Van der pol equation," *Neural, Parllel & Scientific Computations* 7, 1999, 1–16.

[12] V. Strygin and E. Fridman, "Asymptotics of integral manifolds of singularly perturbed differential equations with retarded argument," *Math. Nachr.* 117, 1984, 83–109.