A Generalization of the Widrow's Quantization Theorem

Dorina Isar, Alexandru Isar
Electronics and Telecommunications Faculty, "Politehnica" University
2 Bd. V. Parvan, 1900 Timisoara, Romania

Abstract

The Widrow's quantization theorem is analyzed. This theorem gives the conditions to be satisfied by the probability density function of a random signal for its perfect reconstruction after the quantization process. The disadvantages of this theorem (it's hypotheses are very restrictive) are envisaged and some solutions to decrease the effects of these disadvantages, in the case of different classes of input signals, used in practice, are presented.

1. Introduction

The expression of the probability density function of the random variable O, at the output of a uniform analog to digital converter, ADC, with the quantization step q, excited with the input random variable I is:

$$p_O(x) = \sum_{k=-K}^{K} P(O = o_k) \cdot \delta(x - o_k) \tag{1.1}$$

[1].The condition: $P(O = o_k) = 0$, for $|k| > K$, is natural because at the input of any ADC there are amplitude limiting devices, (the ADC has two saturation zones). Hence, the first order probability density function of the input random signal, I, has compact support. This is the reason why it can be written:

$$p_O(x) = \sum_{k=-\infty}^{\infty} P(O = o_k) \cdot \delta(x - o_k) \tag{1.2}$$

The quantity $P(O = o_k)$ is a function of the variable x, because the output random variable, O, can take the value o_k if and only if the variable x belongs to a certain interval. The quantity $P(O = o_k)$ can be expressed like a convolution between the probability density function of the input random variable, I, $p_I(x)$ and a rectangular pulse, centered in origin, with unitary amplitude and a duration of q. So this probability can be seen like the response of the linear time invariant system with the impulse response:

$$h(x) = \begin{cases} 1 & for \quad |x| \leq \frac{q}{2} \\ 0 & for \quad |x| > \frac{q}{2} \end{cases}$$

to the signal $p_I(x)$. By uniformly sampling of the signal at the output of this system (let the name of this signal be $A(x)$) with the step q, in conformity with the relation (1.2), the signal $p_O(x)$ is obtained. This is the reason why the characteristic function of this signal can be expressed using the relation:

$$\Phi_O(u) = \sum_{k=-\infty}^{\infty} \operatorname{sinc}\left[\frac{q}{2}\left(u - k\frac{2\pi}{q}\right)\right] \Phi_I\left(u - k\frac{2\pi}{q}\right) \tag{1.3}$$

where $\Phi_I(u)$ represents the characteristic function of the random variable at the input of the quantization system. Because an ideal sampling is used, if the hypotheses of the WKS sampling theorem are respected, (the functions $\Phi_V(u)$ and $\Phi_I(u)$ have compact support),[2], then the signal $A(x)$ can be reconstructed from the signal $p_O(x)$. The characteristic function of the random variable V, with the probability density function, $A(x)$, has the expression:

$$\Phi_V(u) = \Phi_I(u) \cdot \operatorname{sinc}\left(\frac{q}{2}u\right)$$

Using the notation:

$$\Phi_U(u) = \operatorname{sinc}\left(\frac{q}{2}u\right)$$

it can be written:

$$\Phi_V(u) = \Phi_I(u) \cdot \Phi_U(u)$$

Because the characteristic function of the random variable V is the product of the characteristic functions of the random variables U and I, the random variable V represents the sum of the random variables U and I and these random variables are independent, [3]:

$$V = U + I \tag{1.4}$$

If $\Phi_V(u)$ has compact support and if a small enough value for the quantization step q is selected, then using $\Phi_O(u)$ the function $\Phi_V(u)$ can be reconstructed with the aid of an ideal low-pass filter. Using $\Phi_V(u)$, the characteristic function $\Phi_I(u)$ can be computed and the probability density function $p_I(x)$ can be perfectly reconstructed. So, if $\Phi_I(u)$ has a compact support and if q is well selected, then from the probability density function of the output random variable can be perfectly reconstructed the probability density function of the input random variable. Hence the quantization process can be inverted. This is the aim of the one dimensional Widrow's quantization theorem, proposed in 1956, [4].

2. The quantization theorem

The enunciation of the quantization theorem, already proved (in the previous paragraph), is the following:

The necessary and sufficient condition that $p_I(x)$ be perfectly reconstructed from $p_O(x)$ is that $\operatorname{supp}\{\Phi_I(u)\}$ have a length of $2u_m$ and to work with a quantization step q such that:

$$\frac{2\pi}{q} > 2u_m \tag{2.5}$$

If the hypotheses of this theorem are satisfied then filtering with an ideal low-pass filter the probability density function $p_O(x)$ we can obtain the characteristic function of **the reconstructed random variable R**:

$$\Phi_R(u) = \Phi_V(u) = \Phi_I(u) \cdot \Phi_U(u)$$

from where we can obtain the probability density function of the input random variable. The moments of the input random variable, I, can be computed using the values of the moments of the reconstructed random variables, R.

3. The computation of the moments of the random variable R

For the beginning it must be observed (on the base of its characteristic function) that the random variable U is uniformly distributed in the interval $\left[-\frac{q}{2}, \frac{q}{2}\right]$. In the following the k-th order moment of the random variable R is computed:

$$\begin{aligned} M\left[R^k\right] &= \frac{1}{j^k}\frac{d^k}{du^k}\Phi_R(u)\mid_{u=0} = \\ &= \sum_{p=0}^{k} C_k^p \left[\frac{1}{j^p}\Phi_I^{(p)}(0)\right]\left[\frac{1}{j^{k-p}}\Phi_U^{(k-p)}(0)\right] = \\ &= \sum_{p=0}^{k} C_k^p M\left[I^p\right] M\left[U^{k-p}\right] \end{aligned} \tag{3.6}$$

For k=1, the last relation becomes:

$$M[R] = M[I] \tag{3.7}$$

So the random variable R has the same average like the input random variable. For k=2, it can be written:

$$M\left[R^2\right] = M\left[S^2\right] + \frac{q^2}{12} \tag{3.8}$$

Hence the power of the random variable R can be computed summing the powers of the random variables I and U. The random variable U can be regarded like a noise, characteristic for the quantization system. This is the reason why using the last relation we can compute the signal to noise ratio at the output of the quantization system:

$$SNR = \frac{\sigma_I^2}{\frac{q^2}{12}} = \frac{12 \cdot \sigma_I^2}{q^2} \tag{3.9}$$

4. The general case

In the following, the hypothesis of compact support for the characteristic function is rejected. In this case the use of the reconstruction low-pass filter is not very useful, because the hypotheses of the sampling theorem are no longer satisfied. This is the reason why we tray to use the output random variable, O, to reconstruct the moments of the input random variable, I. In the following, first, the moments of the random variable O are computed. Using the relation (1.3) it can be written:

$$M\left[O^k\right] = M\left[R^k\right] + \frac{1}{j^k} \sum_{\substack{l=-\infty,\\ l\neq 0}}^{\infty} \sum_{p=0}^{k} C_k^p \Phi_I^{(p)}\left(l\frac{2\pi}{q}\right) \Phi_U^{(k-p)}\left(l\frac{2\pi}{q}\right) \tag{4.10}$$

The error due to the rejection of the compact support hypothesis can be appreciated with the aid of the approximation error of the moments $M\left[R^k\right]$ with the moments $M\left[O^k\right]$. This error is presented in the following relation:

$$\varepsilon_k = \frac{1}{j^k} \sum_{\substack{l=-\infty,\\ l\neq 0}}^{\infty} \sum_{p=0}^{k} C_k^p \Phi_I^{(p)}\left(l\frac{2\pi}{q}\right) \Phi_U^{(k-p)}\left(l\frac{2\pi}{q}\right)$$

If the characteristic function of the input random variable satisfies the hypotheses of the Widrow's quantization theorem then for every k, $\varepsilon_k = 0$. The compact support hypotheses for the functions $p_I(x)$ and $\Phi_I(u)$ are in contradiction because the second function represents the Fourier transform of the first one. The support of $p_I(x)$ is compact because in the signal processing chain there are other operations, before the quantization, implemented with systems with saturated input-output characteristics. **So, the support of $\Phi_I(u)$ can not be compact**. Hence in practice the quantization system can not be perfectly inverted, according to the Widrow's quantization theorem. **In the following are presented other hypotheses, sufficient for the perfect reconstruction of the moments of the input random variable starting from the moments of the output random variable.**

Tacking into account the expressions of the derivatives of the characteristic function of the random variable U we can write:

$$\varepsilon_1 = \frac{q}{2\pi j} \sum_{\substack{l=-\infty,\\ l\neq 0}}^{\infty} \frac{(-1)^l}{l} \Phi_I\left(l\frac{2\pi}{q}\right)$$

So, the relation between the averages of the random variables I and O is:

$$M[O] = M[I] + \frac{q}{2\pi j} \sum_{\substack{l=-\infty,\\ l\neq 0}}^{\infty} \frac{(-1)^l}{l} \Phi_I\left(l\frac{2\pi}{q}\right) \tag{4.11}$$

For $k = 2$ we obtain the following expression of the error:

$$\varepsilon_2 = -\frac{q}{2\pi} \sum_{\substack{l=-\infty,\\ l\neq 0}}^{\infty} \left(\frac{q}{2\pi}\Phi_I\left(l\frac{2\pi}{q}\right)\frac{(-1)^{l+1}}{l^2} + 2\Phi_I'\left(l\frac{2\pi}{q}\right)\frac{(-1)^l}{l}\right)$$

So, the second order moment of the output random variable is:

$$M\left[O^2\right] = M\left[I^2\right] + \frac{q^2}{12} + \varepsilon_2 \tag{4.12}$$

In the following is studied the class of input random variables I, without characteristic function with compact support, satisfying the condition , $\varepsilon_p = 0$, $p = 0$ or more generally $\varepsilon_1 = \varepsilon_2 = 0$. If:

$$\varepsilon_1 = \varepsilon_2 = ... = \varepsilon_k = 0$$

then:

$$\Phi_I^{(p)}\left(l\frac{2\pi}{q}\right) = 0, \quad l \neq 0, \quad p = 0, ..., k-1 \tag{4.13}$$

and the first k order moments of the random variables R and I are the same. So the reconstructed random variable R, obtained when the hypotheses of the Widrow's quantization theorem are satisfied is practically the same with the random variable at the output of the ADC, obtained without satisfying the hypotheses of the Widrow's quantization theorem. So the compact support condition for the characteristic function of the input random variable, requested by the Widrow's quantization theorem, can be rejected without affecting the precision of the reconstruction of the moments of the input random variable. To satisfy the condition $\varepsilon_1 = \varepsilon_2 = 0$ is sufficient to select the input random variable I such that:

$$\Phi_I\left(l\frac{2\pi}{q}\right) = \Phi_I'\left(l\frac{2\pi}{q}\right) = 0, \quad l \neq 0 \tag{4.14}$$

In this case the random variables R and I have the same average and the same variance. These are the moments most frequently used. So, in this case we can reject the compact support condition for the characteristic function of the input random variable without affecting the precision of reconstruction for the moments of the input random variable.

5. Remarkable classes of input signals

In the following table, some examples of input random variables, satisfying the condition (4.14), are presented.

	Characteristic function
$E1:$	$\operatorname{sinc}^2\frac{qu}{2}$
$E2:$	$\operatorname{sinc}\frac{qu}{2}\cdot\operatorname{sinc}\frac{(2K-1)qu}{2}$
$E3:$	$\operatorname{sinc}^2\frac{(2K-1)qu}{2}$
$E4:$	$\prod_{m=1}^{M}\operatorname{sinc}\frac{A_m u}{2}$ where A_m is a multiple of q

Table 1. Some useful classes of input random signals.

For the last example the input random variable is a linear combination of uniform distributed random variables. Their supports must be a multiple of the quantization step. This example includes a lot of random variables. In conformity with the central limit theorem, when $M \to \infty$ the random variable I becomes a normal distributed (gaussian) random variable. This is the reason why we can affirm that in the case of a gaussian random variable the following relations are satisfied in an asymptotic manner:

$$M[O] = M[I] \tag{5.15}$$

$$M\left[O^2\right] = M\left[I^2\right] + \frac{q^2}{12} \tag{5.16}$$

All the elements of the input classes presented in this paragraph generate output random variables for the ADC satisfying the relation (5.15) and (5.16), specific for the Widrow's quantization theorem, without satisfying the hypotheses of this theorem. So the relations (5.15) and (5.16) are satisfied for a large class of input signals. This is the reason why the expression of the output signal to noise ratio in (3.9) is a good estimation of this parameter.

6. The main result

We have already proved a new quantization theorem with the following enunciation:

The necessary and sufficient condition for the perfect reconstruction of the moments of the input random variable, $M\left[I^k\right]$, starting from the moments of the output random variable, $M\left[O^k\right]$, is described in the relation (4.13).

7. Conclusion

The hypotheses of the Widrow's quantization theorem are too restrictive. This is the reason why in this paper are presented, in the paragraph 5, some new results concerning the quantization of the signals members of different classes without satisfying the hypotesis of the Widrow's quantization theorem. To obtain such classes is sufficient to satisfy the conditions presented in the relation (4.13).

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