Quantized Stabilization of Single-input Nonlinear Affine Systems

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Abstract

This paper studies the problem of stabilizing single-input nonlinear affine systems with quantized feedback. We show that, for a single-input nonlinear affine continuous-time system, a stabilizing quantizer can be constructed based on a control Lyapunov function, and a *robustly* stabilizing quantizer can be constructed based on a *robust* control Lyapunov function. We also characterize the coarsest quantizer under certain conditions. The quantized control scheme provides understanding to the problem of how much interaction between the controller and the system dynamics is needed for stabilization.

1 Introduction

In this paper we extend previous results on quantized stabilization of linear systems in [4, 3] to single-input nonlinear affine systems. This research fits into the framework of investigating the complexity of the interaction between controllers and system dynamics, and is useful for studying control systems with communication constraints and computational complexity.

[4] completely characterizes the coarsest quadratically stabilizing quantizer for a single-input linear system, and [3] uses the same Lyapunov-based approach to investigate bounded energy gain performance of quantized single-input linear systems. [6] considers the quantized stabilization of continuous-time linear systems with the introduction of a dwell-time constraint. [2] designs a time-varying uniform quantizer to stabilize a multi-input linear system. [9] extends this design methodology to input-to-state stabilizable nonlinear systems. [10] investigates nonlinear feedback systems and the control effort of quantized feedback. [14, 11] provide the bit-rate needed to stabilize a linear discrete-time system. [5] shows that the quantization of nonlinear systems with symmetry leads to a relatively simple control architecture.

This paper is organized as follows. In Section 2, we introduce the preliminaries of quantization and the Lyapunov-based design approach. In Section 3, we construct quantization for single-input nonlinear affine systems. We show that a countable infinite quantizer that is robustly stabilizing can be derived from a robust CLF (control Lyapunov function). Under certain conditions the coarsest quantizers are given. For a system for which the only available CLF is not robust, we show that the system can be stabilized by a hierarchical quantizer. Finite quantizers are also obtained. In Section 4, we apply the quantization theory to an example.

2 Preliminaries

In this paper we consider the single-input nonlinear affine continuous-time system

$$\dot{x} = f(x) + g(x)u; \quad f(0) = 0. \tag{2.1}$$

where f and g are $\mathcal{C}^1$ functions, $x \in X \subseteq \mathbb{R}^n$, X is the state-space, $u \in \mathcal{U} \subseteq \mathbb{R}$, $\mathcal{U}$ is the admissible control set. We assume the unforced system is unstable.

2.1 Introduction to quantized control

A quantizer is a controller that maps the states of a system into piecewise constant control inputs which take values in an at most countable set. The quantizer only transmits and processes information intermittently and with finite precision, in contrast to a traditional controller which transmits and processes information with infinite precision continuously. Consequently, less interaction is involved in the control process.

Definition 2.1. *A quantizer is a 4-tuple $(q, S, \Omega, \mathcal{U})$ consisting of a map $q : S \to \mathcal{U}$ such that $q(x) = -q(-x)$ and for any $i \in \mathbb{Z}$, $q(x) = u_i$ if $x \in \Omega_i$; a set $S \subseteq X$ containing a neighborhood of the origin; a disjoint partition $\Omega = \{\Omega_i\}_{i=-\infty}^{\infty}$ of S; and a set of admissible control $\mathcal{U} = \{u_i \in \mathbb{R}, i \in \mathbb{Z}\}$. Every Ω_i is called a cell; i is called the index of the cell. With a slight abuse of terminology, q is called the quantizer.*

Logarithmic quantization, as achieved or used in [4, 3, 6], captures the intuition that the farther from the origin the state is, the less precise the control action and knowledge about the location of the state need to be.

Definition 2.2. *A ρ-based logarithmic quantizer is a quantizer $(q, S, \Omega, \mathcal{U})$ with q such that for any $i \in \mathbb{Z}$, $q(x) = u_i$ if $x \in \Omega_i^+$, $q(x) = -u_i$ if $x \in \Omega_i^-$, and $q(x) = 0$ if $x \in \Omega_{zero}$, with Ω being given as*

$$\begin{aligned} \Omega_i^+ &= \{x \in S | \gamma_{i+1} < P'x \leq \gamma_i, \} \quad \forall i \in \mathbb{Z} \\ \Omega_i^- &= \{x \in S | -\gamma_i \leq P'x < -\gamma_{i+1}, \} \quad \forall i \in \mathbb{Z} \\ \Omega_{zero} &= \{x \in S | P'x = 0\} \end{aligned} \tag{2.2}$$

and with $\mathcal{U} = \{\pm u_i | u_{i+1} = \rho u_i, i \in \mathbb{Z}\} \cup \{0\}$, where $0 < \rho < 1$ is called the base, $P \in \mathbb{R}^n$ is a constant vector, $\gamma_{i+1} = \rho \gamma_i$, $i \in \mathbb{Z}$, γ_0 is assumed to be 1 without loss of generality. For cell Ω_i^+ (or Ω_i^-, Ω_{zero}), the index is i^+ (or i^-, zero, respectively).

For nonlinear systems, it is often convenient to consider semi-log quantizers, which have the major properties of log quantizers.

Definition 2.3. *A ρ-based semi-logarithmic quantizer is a quantizer as defined in Definition 2.2 except that the linear function $P'x$ is replaced by a smooth function $p(x) : S \to \mathbb{R}$ with $p(0) = 0$.*

Note first that when designing a semi-log quantizer, we need to specify S, ρ, $p(x)$ and u_0. Second, since both the control value u and the partition $p(x)$ follow a logarithmic law with the same base, the graph of the quantizer in the $p(x)$-u plane is self-similar with similarity ratio ρ. Thus, in practice it is easy to do calculation online or store the relevant data in memory. See Fig. 1. Third, the cells become larger and larger when $p(x)$ is farther away from 0, as for log quantizers. Fourth, whenever the state x is approaching the boundary of a cell, the corresponding point in the $p(x)$-u plane is either approaching the line $u = k_1 p(x)$, or approaching the line $u = k_2 p(x)$. In the $X \times \mathcal{U}$ space, $u = k_1 p(x)$ and $u = k_2 p(x)$ are two manifolds, which are called the *triggering manifolds*.

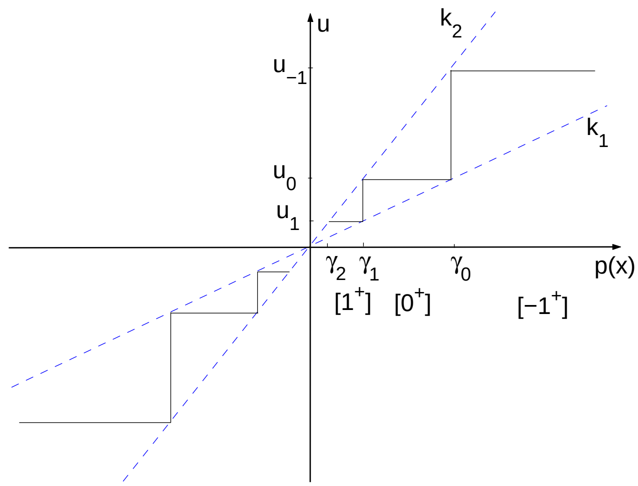

Figure 1: The graph of semi-log partition in the $p(x)$-u plane. Each number in brackets is the index of the cell. For any x s.t. $p(x) \in (\gamma_1, \gamma_0]$, index 0^+ is transmitted and u_0 is used as the control input.

For log quantizers, the two triggering manifolds are simply two subspaces in the $X \times \mathcal{U}$ space. Consequently, every cell is rectilinear, which makes its implementation easy. Note that once a semi-log quantization is given, we can always define a coordinate transformation $z = T(x)$ so that the quantization in the new coordinates is logarithmic. See Fig. 2 for log and semi-log partitions in a 2-dimensional state-space.

A system with a (semi-)log quantizer can be seen as an automaton. The automaton has a countable infinite number of states, with a fixed output (i.e., the control input of the system) assigned to each of them. Each state of the automaton is associated with one cell in the system's state-space. An instantaneous transition to a different state takes place if x crosses the boundary of the cell, and the new state of the automaton is decided by the position of x, i.e., the index of the cell that x is entering. As the system evolves continuously, the automaton evolves at discrete instants of time, and generates corresponding control inputs. Fig. 3 illustrates the state transition of an automaton.

Next, we introduce finite quantizers.

Definition 2.4. *A finite quantizer (of order N) is a quantizer q with* $\Omega = \{\Omega_i\}_{i=-N+1}^{N-1}$*, and*

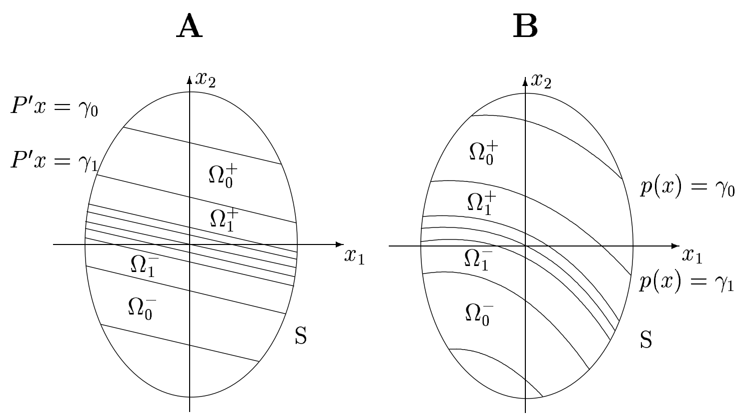

Figure 2: Examples of log and semi-log partitions in 2 D state-space. A shows a log quantizer defined on S, and B shows a semi-log quantizer defined on S. Log partition has straight boundary lines, and for semi-log they are curves.

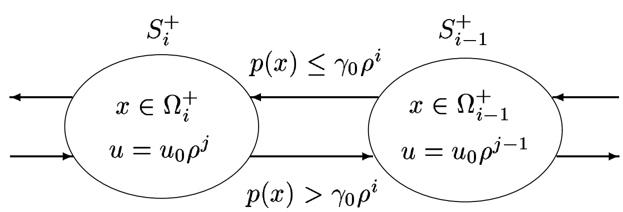

Figure 3: A hybrid automaton for a semi-log quantizer. S_i^+ denotes the state of the automaton associated with cell Ω_i^+, etc.

$\mathcal{U} = \{u_i \in \mathbb{R} | u_i = -u_{-i}, i = -N+1, \cdots, 0, \cdots, N-1\}$.

We define the density of a quantizer as follows.

Definition 2.5. *For a quantizer q of system (2.1), let $0 < \epsilon \leq 1$ and $\#q[\epsilon]$ denote the number of levels that q has in the interval $[\epsilon, \frac{1}{\epsilon}]$. Define*

$$\eta_q = \limsup_{\epsilon \to 0} \frac{\#q[\epsilon]}{-\ln \epsilon}.$$

η_q is called the quantization density of q. For two quantizers f and g, f is said to be coarser than g if $\eta_f < \eta_g$.

A ρ-based semi-log quantizer q defined on $\mathbb{R}^n$ has density $\eta_q = \frac{2}{\ln \frac{1}{\rho}}$. A ρ-based semi-log quantizer q defined on a compact set in $\mathbb{R}^n$ has density $\eta_q = \frac{1}{\ln \frac{1}{\rho}}$. A finite quantizer has density zero. Two quantizers which are only different in a finite number of levels have the same density.

2.2 Introduction to the Lyapunov-based design

In this paper we will construct quantizers for system (2.1) based on the availability of a CLF for (2.1). In this section we introduce the concept of an RCLF (robust CLF). We will see that the availability of an RCLF guarantees the existence of a finite density stabilizing quantizer. The quantizer design method developed for RCLF's extends to a method for CLF's when only a CLF is available. In this paper smoothness is assumed for all CLF's.

Definition 2.6. *$(V(x), \alpha)$ is called an RCLP (robust control Lyapunov pair) for system (2.1) on a compact set $S \subseteq X$ containing a neighborhood of 0 if $\alpha > 0$, $V(x)$ is a CLF for (2.1) on S, and there exists some admissible control u_x for each $x \neq 0$ in S, such that*

$$\alpha^2 u_x^2 + L_g V(x) u_x + L_f V(x) < 0. \tag{2.3}$$

The $V(x)$ in the above definition is called an RCLF for (2.1) on S. Here $L_g V(x) = \frac{\partial V}{\partial x} g$, and $L_f V(x) = \frac{\partial V}{\partial x} f$. For simplicity, we always assume without loss of generality that S is a closed level set of $V(x)$.

The RCLF is so called since it guarantees certain robustness, as shown in Appendix 6.1. In addition, it can be shown that, for a given RCLF $V(x)$, RCLP $(V(x), \alpha_1)$ guarantees more robustness than $(V(x), \alpha_2)$ if $\alpha_1 > \alpha_2$. Therefore, we call α the *robustness level.* Several classes of nonlinear affine systems, such as linear systems and linearizable systems, admit RCLF's.

Equation (2.3) describes a reasonable requirement for many control systems: it requires V to decrease, and it gives penalty to using large control. This renders finite gain at the origin [1]. In [12, 7] a generalized form of equation (2.3) is used to achieve optimality and robustness to a class of uncertainties.

Define $h(x) = \frac{(L_g V)^2}{4 L_f V}$ on the set $S_f = \{x \in S | L_f V(x) > 0\}$. Let $\alpha_M^2 = \inf h(x)$, and $\alpha_S^2 = \liminf_{x \to 0} h(x)$. Then an RCLF has the following properties.

Lemma 2.1. *Suppose V is a CLF for system (2.1) on S. The following statements are equivalent:*

(1) $V(x)$ is an RCLF for system (2.1) on S.

(2) V is such that $u = k L_g V(x)$ is stabilizing for some constant $k < 0$.

(3) V is such that $\alpha_M^2 > 0$.

(4) V is such that $\alpha_S^2 > 0$.

Condition (2) says that $V(x)$ is an RCLF if and only if *domination redesign* (cf. [7]) is applicable. Condition (3) says that the problem of checking when a CLF is an RCLF is reduced to solving a constrained optimization problem. Condition (4) says that V is an RCLF if and only it has a certain limiting property around the origin. These conditions may be difficult to check. For checking when a CLF is an RCLF, the following lemma provides a sufficient condition [7]:

Lemma 2.2. *Suppose $V(x)$ is a CLF for system (2.1) on S. Suppose further the first nontrivial terms in the Taylor expansions of $f(x)$, $g(x)$, and $V(x)$ have degrees d_f, d_g, and d_V, respectively, and let $f_h(x)$, $g_h(x)$, and $V_h(x)$ denote these nontrivial terms, respectively. If V_h is a CLF for the system $\dot{x} = f_h(x) + g_h(x)u$ and if $d_V \leq d_f - 2d_g$, then $V(x)$ is an RCLF for system (2.1) on S.*

Once we have verified that a CLF is indeed an RCLF, then $(V(x), \alpha)$ is an RCLP if $0 < \alpha^2 < \alpha_M^2$.

We define the coarsest quantizer, or the quantizer with the smallest density, as follows.

Definition 2.7. *Given an RCLP $(V(x), \alpha)$ for system (2.1) on S, $\alpha^2 < \alpha_M^2$, let $\mathcal{Q}_\alpha(V)$ denote the set of all quantizers $q(x)$ such that for any $x \in S$, $x \neq 0$,*

$$\alpha^2 (q(x))^2 + L_g V(x) q(x) + L_f V(x) < 0. \tag{2.4}$$

A quantizer q is said to be the coarsest for $(V(x), \alpha)$ if $q = \arg\inf_{g \in \mathcal{Q}_\alpha(V)} \eta_g$.

3 Quantized single-input nonlinear affine systems

3.1 Main results

In this section, we are interested in the following questions. What conditions imply quantized (robust) stabilization of system (2.1)? What is the smallest possible density of the (robustly) stabilizing quantizer? How do we construct such a quantizer? For the concepts of stability and robustness used in this paper, we refer to Appendix 6.2.

Our main results are as follows.

Theorem 3.1. *(a) If system (2.1) admits a CLF on S, then (2.1) can be stabilized by a quantizer with countable infinite levels, or practically stabilized by a finite quantizer.*

(b) If system (2.1) admits an RCLF on S, then (2.1) can be robustly stabilized by a finite density quantizer, or stabilized by a finite quantizer.

Theorem 3.2. *Suppose $(V(x), \alpha)$ is an RCLP for system (2.1) on S. If $\alpha_S^2 < +\infty$, then the coarsest quantizer q^* for $(V(x), \alpha)$ has density $\eta^* = \frac{1}{\ln \frac{1}{\rho^*}}$, where $\rho^* = k_1/k_2$, $k_1 = \frac{-1+\sqrt{1-\alpha^2/\alpha_S^2}}{2\alpha^2}$, $k_2 = \frac{-1-\sqrt{1-\alpha^2/\alpha_S^2}}{2\alpha^2}$. If $\alpha_S^2 = +\infty$, then the coarsest quantizer q^* for $(V(x), \alpha)$ has density $\eta^* = 0$.*

Theorem 3.1 allows for the existence of a variety of stabilizing quantizers. Theorem 3.2 provides the density of the coarsest quantizer for a given RCLP. In the remainder of this section, we will first construct quantizers based on a given RCLF, and then on a given CLF.

3.2 Quantizers for RCLF's

We want to stress that η^* provided in Theorem 3.2 may not be an achieved infimum over all robustly stabilizing quantizers for a given RCLP. In other words, in some cases $q^* \bar{\in} \mathcal{Q}_\alpha(V)$ ($\bar{\in}$ meaning "does not belong"). The following lemma shows when the density η^* is achievable.

Lemma 3.1. *$q^* \in \mathcal{Q}_\alpha(V)$ if and only if there is some $a > 0$ such that $\alpha_S^2 < h(x)$ for all x in $S_{f_a} = \{x \in S_f | V(x) \leq a\}$.*

Next, we will construct the coarsest quantizer q^* if η^* is achievable, and in case η^* is not achievable, we will construct a quantizer q_ϵ with density $\eta^* + \epsilon$ for any given $\epsilon > 0$.

Notice that if $\Omega_Z \triangleq \{x \in S | L_f V(x) < -\delta \|x\|^2, \delta > 0\}$ is nonempty, then we only need to define quantizers on $\Omega_{NZ} = S \setminus \Omega_Z$ since on Ω_Z we can use zero control input to hold (2.4).

3.2.1 Construction of the coarsest quantizer q^*

Suppose η^* is achievable. Let $S_a = \{x \in S | V(x) \leq a\}$, i.e., the smallest closed level set of $V(x)$ containing S_{f_a}. Then we can use a finite number of control values to drive the state from $S \setminus S_a$ into S_a (see Theorem 3.1(b)), and then focus on a smaller invariant set S_a, on which the coarsest quantization turns out to be semi-logarithmic. Since a finite number of control values do not affect the quantizer density, we know the density is determined by the quantization defined on S_a.

Proposition 3.1. *Suppose $(V(x), \alpha)$ is an RCLP for system (2.1) on S, and there is some $a > 0$ such that $\alpha_S^2 < h(x)$ for all x in S_{f_a}. Then there exists a robustly stabilizing quantizer $q^* \in \mathcal{Q}_\alpha(V)$ with density η^*. q^* has a finite number of cells on $S \setminus S_a$, and is ρ^*-based semi-logarithmic on $S_a \cap \Omega_{NZ}$ with $p(x) = L_g V(x)$ and $u_0 = k_1 \gamma_0$.*

Note that the smaller the α^2, the coarser the quantization is, but the less robust the closed-loop system is. Although Proposition 3.1 gives the coarsest quantizer under some conditions, we want to remark that for a continuous-time system, quantization density is only a partial measure of the complexity of the interaction between the quantizer and the system dynamics, in contrast to discrete-time systems [4]. Other quantities related to the information processing and transmission, such as average switching time, are also important [10].

3.2.2 Construction of quantizer q_ϵ

Now we consider the case that η^* is not achievable. Given any $\epsilon > 0$, let $\eta_\epsilon = \eta^* + \epsilon$, $\rho_\epsilon = e^{-\frac{1}{\eta_\epsilon}}$, $\alpha_\epsilon^2 = \frac{\alpha^2}{1-(\frac{1-\rho_\epsilon}{1+\rho_\epsilon})^2}$, $S_{f\epsilon} = \{x \in S_f | h(x) \geq \alpha_\epsilon^2\}$, and S_ϵ be the smallest closed level set of $V(x)$ containing $S_{f\epsilon}$. Then we can use a finite number of control values to drive the state from $S \setminus S_\epsilon$ into S_ϵ, and on S_ϵ we can construct a robustly stabilizing quantizer with density η_ϵ.

Proposition 3.2. *Suppose $(V(x), \alpha)$ is an RCLP for system (2.1) on S. Then for any given $\epsilon > 0$, there exists a robustly stabilizing quantizer $q_\epsilon \in \mathcal{Q}_\alpha(V)$ with density $\eta_\epsilon = \eta^* + \epsilon$. q_ϵ has a finite number of cells on $S \setminus S_\epsilon$, and is ρ_ϵ-based semi-logarithmic on $S_\epsilon \cap \Omega_{NZ}$ with $p(x) = L_g V(x)$ and $u_0 = k_1 \gamma_0$ where $k_1 = \frac{-1+\sqrt{1-\alpha^2/\alpha_\epsilon^2}}{2\alpha^2}$.*

3.3 Quantizers for systems with CLF's

For a CLF that is not an RCLF, we have $\alpha_M^2 = \alpha_S^2 = 0$ on S. Some of these systems can be stabilized by a finite density quantizer, such as $\dot{x} = x^3 + x^2 u$. However, others may need an infinite density quantizer to stabilize, such as $\dot{x} = x + x^2 u$. In this subsection, we present an approach to design a stabilizing quantizer (possibly with infinite density) based on a given CLF. Although such a quantizer may be difficult to implement, it will be seen later that its finite truncation leads to practical stabilization, and hence it is useful.

We first partition $\Omega_{NZ} \setminus \{0\}$ into disjoint subsets $\{K_m\}_{m=1}^\infty$ with $0 \bar{\in} \mathrm{cl}(K_m)$ for all m. Let $\alpha_{Mm}^2 = \inf_{x \in K_m \cap S_f} h(x)$. Obviously, we have $\alpha_{Mm}^2 > 0$ for all m. Then we can define a semi-log quantizer q_m on each K_m. If the state is in K_m, then q_m is employed. If the state is driven outside of K_m into K_{m+1}, then we switch to the quantizer q_{m+1}. Each q_m makes V decrease and finally sends the state to the origin. This leads to a hierarchical quantization structure.

Proposition 3.3. *Suppose V is a CLF for system (2.1) on S. System (2.1) can be semi-globally stabilized to the origin by a hierarchical quantizer. Level 1 quantization is a partition of $\Omega_{NZ} \setminus \{0\}$ by disjoint sets $\{K_m\}_{m=1}^\infty$ with $0 \bar{\in} \mathrm{cl}(K_m)$ for all m. Level 2 quantization is obtained by defining a ρ_m-based semi-log quantizer q_m on each set K_m with $p(x) = L_g V(x)$ and $u_0 = k_{1m}\gamma_0$, where $\rho_m = k_{1m}/k_{2m}$, $k_{1m} = \frac{-1+\sqrt{1-\alpha^2/\alpha_{Mm}^2}}{2\alpha^2}$, and $k_{2m} = \frac{-1-\sqrt{1-\alpha^2/\alpha_{Mm}^2}}{2\alpha^2}$.*

Here $\mathrm{cl}(\cdot)$ denotes the closure. Level 1 partition is normally given by the level surfaces of $V(x)$, $L_g V(x)$, $\|x\|$, etc. Proposition 3.3 implies that a general control architecture can be built for system (2.1) if (2.1) admits a CLF. This architecture is a 3-level hierarchical quantizer. System (2.1) with the quantizer can be seen as a hierarchical automaton; refer to Fig. 4.

3.4 Further discussion

In this part we briefly discuss finite quantizers, the relation between smooth feedback and quantized feedback, and chattering-free quantizers.

We can show that no finite quantizer can be robustly stabilizing. In order to obtain a finite quantizer, we need to relax the requirement of robust stabilization. In fact, a finite truncation of the infinite quantizer in Proposition 3.1 guarantees stabilization instead of robust stabilization.

A finite truncation of the quantizer defined in Proposition 3.1 is obtained as follows. For some $j \in \mathbb{Z}$, let $\Omega_*^+ = \{x \in \Omega_{NZ} | 0 < L_g V(x) \leq \gamma_j\}$, and $\Omega_*^- = \{x \in \Omega_{NZ} | 0 > L_g V(x) \geq$

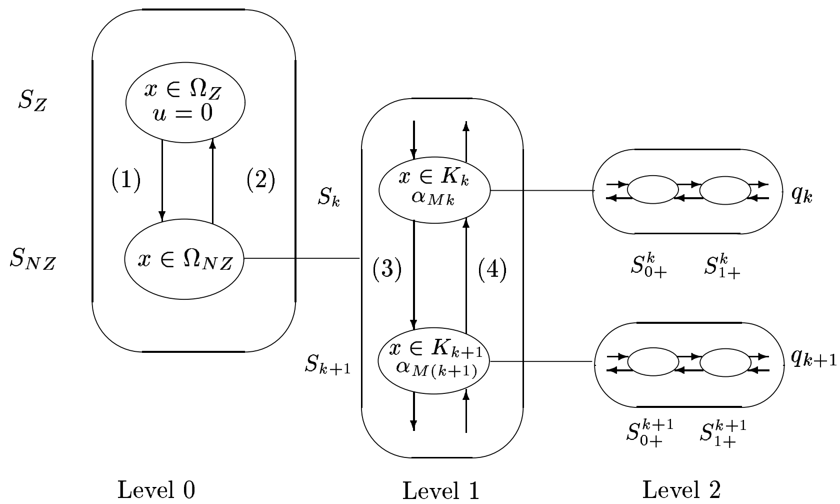

Figure 4: 3-level hierarchical automaton for Proposition 3.3. The state S_{NZ} in Level 0 is the automaton in Level 1. The state S_k in Level 1 is the automaton q_k in Level 2. If Level 1 quantization is decided by the level surfaces of $\|x\|$, then the logic conditions for state transition are: (1) $L_f V(x) \geq -\delta\|x\|^2$, (2) $L_f V(x) < -\delta\|x\|^2$, (3) $\|x\|^2 < c_m$, (4) $\|x\|^2 \geq c_m$.

$-\gamma_j\}$, and use $u^* = k_1\gamma_j$ in Ω_*^+, $-u^*$ in Ω_*^-. For any $i < j$, let $\Omega_i^\pm$ and the corresponding u be as in Proposition 3.1.

Corollary 3.1. *The finite truncation of the quantizer in Proposition 3.1 semi-globally stabilizes (2.1) to the origin.*

This corollary says that the finite truncation of a robustly stabilizing infinite quantizer is still stabilizing with the loss of robustness and loss of finite gain in a small neighborhood of the origin. Similarly, we can show that the finite truncation of the quantizer defined in Proposition 3.2 leads to stabilization instead of robust stabilization, and the finite truncation of the quantizer defined in Proposition 3.3 leads to practical stabilization instead of stabilization.

Based on these results we can establish, using Artstein's Theorem (see [13]), the following conclusion: if a nonlinear affine system can be stabilized by smooth feedback (possibly discontinuous at the origin), then it can be stabilized by quantized feedback with countable infinite levels, or practically stabilized by quantized feedback with only finite levels.

Finally, we would like to mention that chattering, or infinitely fast switching, may occur in quantized control as a result of discontinuous RHS vector fields. Chattering can be physically harmful to systems. It can be shown that the quantizers designed in this section can be made chattering-free and lead to practical stability by applying *switching control with dwell time* [2, 6]. In this approach, switching logic with a fixed dwell time is used to guarantee finite switchings in finite time.

4 An example: quantized controller for a vehicle

We consider a simplified vehicle model in a plane as follows:

$$\begin{aligned} \dot{x_1} &= -\cos x_3 \\ \dot{x_2} &= \sin x_3 \\ \dot{x_3} &= u \end{aligned} \tag{4.1}$$

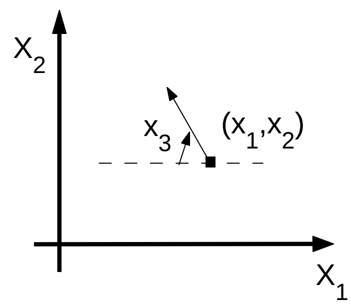

Figure 5: A simplified kinematic model of a vehicle.

where x_3 is the steering angle, u is the steering angular velocity, and the linear velocity is 1. We will design a quantizer so that the vehicle can track the x_1 axis in the x_1-x_2 plane and point due west (left). (With only a coordinate transformation, the designed quantizer can be used to track any straight line in the plane.) Since there is no requirement on x_1, we focus only on the dynamics of x_2 and x_3. Once x_2 and x_3 are stabilized, the vehicle is running along the desired trajectory. It is easy to verify that $V = x_2^2 + x_3^2 + x_2 x_3$ is an RCLF for the dynamics of x_2 and x_3, and $\alpha_M^2 = \alpha_S^2 = 3/4$. By Proposition 3.1, we know that the dynamics of x_2 and x_3 are robustly stabilized by a log quantizer. Level 0 partition is given by $\Omega_Z = \{x | \sin x_3(2x_2 + x_3) < -\delta(x_2^2 + x_3^2)$ and $\Omega_{NZ} = \{x | \sin x_3(2x_2 + x_3) \geq -\delta(x_2^2 + x_3^2)\}$ for some $\delta > 0\}$. Ω_{NZ} is logarithmically partitioned by $p(x) = x_2 + 2x_3$ (Level 1 partition). Finite truncation of this quantizer is stabilizing.

Fig. 6 is a sample trajectory using the quantizer given above. The vehicle is running in the x_1-x_2 plane. A 2-level quantization is defined on the x_2-x_3 plane. The dashed line is the desired trajectory in the x_1-x_2 plane. The trajectory has been plotted in x_1-x_2-x_3 space, as well as projections in the x_1-x_2 and x_2-x_3 planes. Stars represent the switching points. We can see from the figure that the vehicle follows a natural trajectory to reach the desired trajectory and then goes along it. Interaction between the quantizer and the vehicle only exists at the star points.

5 Conclusion

In this paper, we have extended the results on quantization of linear systems to single-input nonlinear affine systems, showing that a single-input nonlinear affine system can be

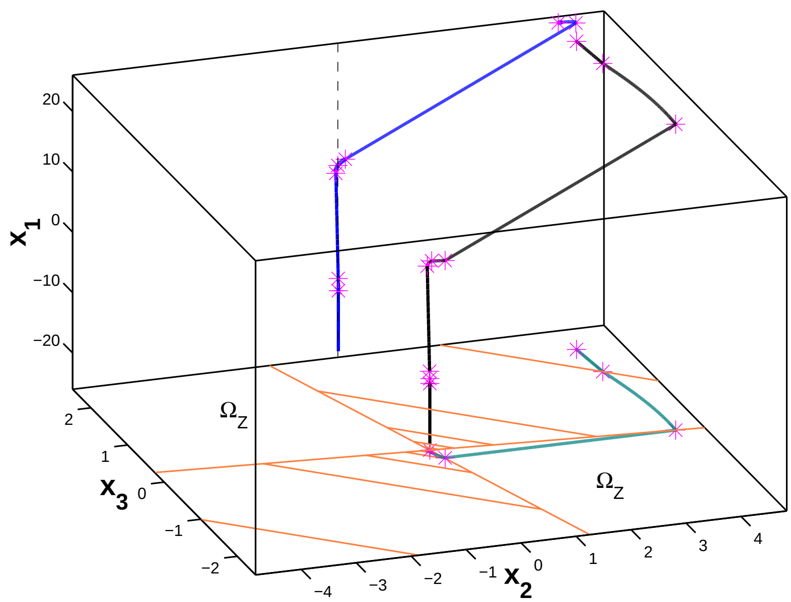

Figure 6: A sample trajectory of the vehicle's line tracking.

(robustly) stabilized by quantized feedback if it admits a (robust) CLF. We have shown that under certain conditions the coarsest quantizer follows a semi-logarithmic law. The designed quantizers in the closed-loop can be viewed as (hierarchical) hybrid automata. The quantized control strategy leads to a general control architecture for all single-input nonlinear affine systems with CLF's. This control architecture is helpful in reducing the interaction between the controller and the system being controlled. We have designed a quantized controller for a simple vehicle using the obtained results.

6 Appendix

6.1 RCLF and UCLF

Consider a single-input control system under a persistently acting disturbance

$$\dot{x} = F_d(x, u, d) \tag{6.1}$$

where $x \in X, u \in \mathcal{U}$ are defined as before, F_d is continuous, the disturbance $d(\cdot)$ is a measurable function taking values in D, D is a compact set of admissible disturbance, $d_M = \max_{d \in D} |d|$.

A function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is said to be *positive definite* if $V(x) > 0$ for all $x \neq 0$, and $V(0) = 0$. It is said to be *proper* if $\{x | V(x) \leq a\}$ is compact for all $a > 0$. A smooth, positive definite and proper function $V : X \to \mathbb{R}_{\geq 0}$ is called a UCLF (uniform CLF; refer to [8]) if there exists a continuous positive definite function $W : X \to \mathbb{R}_{\geq 0}$, and for any bounded set $E \subseteq X$, there is some compact set of admissible control $\mathbb{U}$ such that

$$\min_{u \in \mathbb{U}} \max_{d \in D} \langle \nabla V, F_d(x,u,d) \rangle < -W(x) \quad \forall x \in E, x \neq 0. \tag{6.2}$$

Roughly speaking, a UCLF is a CLF whose derivative can be made negative pointwise by the choice of control value for any admissible disturbance d. In the next lemma we show that a CLF for the undisturbed system

$$\dot{x} = F(x,u) \tag{6.3}$$

with certain properties is also a UCLF for the system with a small enough persistently acting disturbance

$$\dot{x} = F(x,u) + G(x,u)d \tag{6.4}$$

where $F(0,0) = G(0,0) = 0$, F and G are continuous. Furthermore, we assume $\|G(x,u)\|/u^2$ is bounded by a constant c on the set $X \times U$. This assumption implies that the effect of the disturbance d cannot dominate the control input u; otherwise the system may not be able to be controlled.

Lemma 6.1. *Suppose V is a CLF for the undisturbed system (6.3) on S, and $V_M = \max_{x \in S} \|\frac{\partial V}{\partial x}\|$. V is a UCLF for (6.4) on S if $\alpha^2 > cV_M d_M$, and if there exists some $u_x \in U$ for each $x \neq 0$ in S, such that*

$$\alpha^2 u_x^2 + \langle \nabla V, F(x, u_x) \rangle < 0. \tag{6.5}$$

This lemma says in essence that if the derivative of a CLF for an undisturbed system can be made negative enough pointwise by the choice of control input, then it is a UCLF for a disturbed system if the disturbances are small enough. Notice that a larger $\frac{\alpha}{V_M}$ implies more robustness of the closed-loop system for a given V. If we normalize V_M to be 1, then α can measure the robustness of the closed-loop system, and we call it the *robustness level* in this paper. The V just described is called an RCLF. Definition 2.6 defines this precisely for the nonlinear affine case. Therefore the RCLF defined in this paper is in fact the UCLF defined in [8].

6.2 Discontinuous systems, stability, and robustness

Quantized control is a kind of discontinuous control. Once a discontinuous feedback control $k(x)$ is employed, the existence and uniqueness of solutions, the notions of stability and robustness, and related theorems need to be reexamined or restated. In this paper, the solutions to a quantized control system as well as the stability and robustness are to be interpreted according to [13, 8]. It can be shown that quantizers obtained in this paper guarantee the existence of solutions, but not the uniqueness. Here we only present a lemma connecting (robust) CLF's to (robust) stabilization of discontinuous systems, which follows directly from [13, 8].

Lemma 6.2. *(a) Suppose V is a CLF for system (6.3). Then if $k(x)$ is such that*

$$\langle \nabla V, F(x, k(x)) \rangle < 0 \quad \forall x \neq 0 \tag{6.6}$$

then $k(x)$ is a stabilizing feedback.
(b) Suppose V is an RCLF for system (6.3). Then if $k(x)$ is such that for some $\alpha > 0$,

$$\alpha^2 k^2(x) + \langle \nabla V, F(x, k(x)) \rangle < 0 \quad \forall x \neq 0 \tag{6.7}$$

then $k(x)$ is a robustly stabilizing feedback under the presence of persistently acting disturbance $d(t)$, measurement errors $e(t)$, and external disturbances $w(t)$; i.e., $k(x)$ stabilizes system

$$\dot{x} = F(x, k(x + e(t))) + G(x, k(x + e(t)))d(t) + w(t). \tag{6.8}$$

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