Stabilizing quantized feedback with minimal information flow: the scalar case

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Abstract

A state feedback with finitely many quantization levels yields only the so called practical stabilization, namely the convergence of any initial state belonging to a bigger bounded region into another smaller target region of the state space. The ratio between the measure of the starting region and the target region is called contraction of the closed loop system. In the analysis of the performance of a stabilization strategy based on a quantized state feedback two parameters play a central role: the number of quantization levels used by the feedback and the convergence time of the closed loop system.

In this paper we propose a definition of optimality for a quantized stabilization strategy. This definition is based on how the number of quantization levels and the convergence time grow with the contraction. Then, we analyze the performance and prove the optimality of three different stabilizing quantized feedbacks strategy for scalar linear systems.

Keywords: Stability, stabilization, communication constraints, quantized feedback, chaotic control.

1 Introduction

In recent years a certain interest has been developed on the control problems in which communication constraints have to be considered. Systems with communication constraints can be considered as instances of of hybrid systems in which particular attention is devoted to the data flow. Control problems for these systems are very difficult to solve and a general theory seems still far to be developed. Some contributions have been given in [1, 3, 4, 5, 7, 8, 9, 10, 11, 12].

Discrete time systems with quantized feedback can be seen as particularly simple cases of dynamical systems in which the control requires a finite information flow. This class of systems can be analyzed in more detail even though they are nonlinear systems with wild behavior. In this set up the information flow has to be quantified in terms of the number of quantization levels of the feedback function. The problem in this context can be formulated as follows: What is the minimal information flow required for fulfilling a certain control objective? In control theory stabilization is considered the simplest control objective for which the previous question becomes the following: What is the minimal information flow required for stabilizing a discrete time unstable systems?

In this paper we will show that this question does not makes sense if we do not evaluate also the performance of the closed loop system. We will show that there are different stabilizing quantized feedback strategies requiring different information flows but providing closed loop systems with different stability performances. Stability performance can be measured in different ways. In this contribution we choose to evaluate stability performance in terms of the convergence time. Other choices are possible but they are not explored here. The main contribution of this paper is to propose a methodologic framework in which a quantized stabilization feedback strategy and its optimality can be defined and to prove the optimality of three different strategies. The first is based on the approximation of a deadbeat controller by a uniform quantizer [3]. The second strategy is a Lyapunov based quantized stabilization method [4]. The third one exploits the chaotic behavior of closed loop systems with a quantized feedback [5]. In this paper our investigations are limited to linear scalar systems.

2 Scalar linear systems with quantized feedback and quantized feedback strategies

Consider the following discrete-time, one-dimensional linear model

$$x_{t+1} = ax_t + u_t \tag{2.1}$$

where $a \in \mathbb{R}$. Let $k : \mathbb{R} \to \mathbb{R}$ be a piecewise constant function with only finitely many discontinuities. If we use k as a static feedback in (2.1), we obtain the closed loop system

$$x_{t+1} = \Gamma(x_t), \tag{2.2}$$

where $\Gamma(x) := ax + k(x)$ is a piecewise affine map. Autonomous systems like (2.2) in which Γ is piecewise affine can exhibit a very wild behavior. Their dynamical properties have been extensively studied in the past [6, 2]. It is obvious that in this case only a "practical stability" can be obtained as detailed in the following definitions.

Definition: Stability and almost stability. Given two intervals $J \subseteq I$, we say that $\Gamma : \mathbb{R} \to \mathbb{R}$ is (I, J)-*stable* if for every $x_0 \in I$, there exists an integer $T \geq 0$ such that $x_t \in J$ for every $t \geq T$. We say that Γ is *almost* (I, J)-*stable* if the convergence to J as defined above occur for almost all $x_0 \in I$, with respect to the normalized Lebesgue measure λ_I. A quantized feedback map $k : \mathbb{R} \to \mathbb{R}$ is said to be *(almost)* (I, J)-*stabilizing* if the corresponding closed loop map Γ is (almost) (I, J)-stable.

It is clear that the above definitions of stability and almost stability only depend on the restriction of Γ to I and so we can assume that Γ is defined only on I. The first entrance time function

$$T_J : I \to \mathbb{N} \cup \{+\infty\}$$

is defined by

$$T_J(x) = \inf\{t \in \mathbb{N} \mid \Gamma^t x \in J\}, \tag{2.3}$$

and by $T_J(x) = +\infty$ if $\Gamma^t x \notin J$ for all t. The map T_J is always finite exactly when we have stability, while it is almost surely finite when we have almost stability. The performance of a stabilizing quantized feedback will be evaluated through the expected value $\mathbf{E}(T_J)$ of T_J with respect to the normalized Lebesgue measure $\mathbf{E}(T_J)$ which coincides with

$$\mathbf{E}(T_J) = \sum_{n=1}^{\infty} n\lambda_I[T_J = n] = \sum_{n=1}^{\infty} \lambda_I[T_J \geq n],$$

where $\lambda_I[T_J = n]$ and $\lambda_I[T_J \geq n]$ are shorthand notations for $\lambda_I[\{x \in I | T_J(x) = n\}]$ and $\lambda_I[\{x \in I | T_J(x) \geq n\}]$, respectively. Optimality of a feedback strategy will be evaluated in terms of the expected convergence time of the closed loop system with a given number N of quantization intervals. This optimality will be defined with respect to the asymptotic dependence of the parameters $\mathbf{E}(T_J)$ and N on the contraction rate $C := \lambda_I(J)^{-1}$. For this reason we define a quantized feedback strategy as a sequence of controllers instead of a single control.

Definition: Stabilizing quantized feedback strategy. Given an interval I and a family of intervals $\{J_C | C \in \mathbb{R}_+\}$, such that $J_C \subseteq I$ for all $C \geq 1$ and such that $C = \lambda_I(J_C)^{-1}$, a family of quantized feedback maps

$$\mathbb{K} := \{k_C : I \to I \mid C \geq 1\}$$

is said to be a *(almost) stabilizing quantized feedback strategy* if k_C is (almost) (I, J_C)-stabilizing for every $C \in \mathbb{R}_+$. The number of quantization intervals and the expected convergence time of the quantized feedback k_C will be denoted by the symbols $N(C)$ and $T_m(C)$ respectively.

Definition: Optimality of a stabilizing quantized feedback strategy. We say that a stabilizing quantized feedback strategy $\mathbb{K} := \{k_C : I \to I \mid C \geq 1\}$ is optimal if for any stabilizing quantized feedback strategy $\mathbb{K}' := \{k'_C : I \to I \mid C \geq 1\}$ the following two conditions hold true:

(i) There exists a positive constant K such that

$$\limsup_{C\to\infty} \frac{N'(C)}{N(C)} \leq 1 \qquad \Longrightarrow \qquad \liminf_{C\to\infty} \frac{T'_m(C)}{T_m(C)} \geq K.$$

(ii) There exists a positive constant K such that

$$\limsup_{C\to\infty} \frac{T'_m(C)}{T_m(C)} \leq 1 \qquad \Longrightarrow \qquad \liminf_{C\to\infty} \frac{N'(C)}{N(C)} \geq K.$$

3 Three stabilizing quantized feedback strategies

We will present now three different stabilizing quantized feedback strategies whose performance will be analyzed in the sequel.

Consider the linear discrete time system (2.1) where $|a| > 1$. Let $I = [-1, 1]$ and $J = [-1/C, 1/C]$, with $C \geq 1$. We want to stabilize it through a quantized state feedback, i.e. we want to find a quantized feedback map k such that the closed loop system (2.2) drives (almost) any initial state $x_0 \in I$ into a state evolution which, after a transient, enters the interval J. There exists several solutions to this problem.

3.1 Deadbeat quantized feedback strategy

The first solution is simply to approximate the 1-step deadbeat controller $k(x) := -ax$ with its quantized version, i.e., by a uniform quantized function $k(x)$ such $k(x)$ such $-ax - 1/C \leq k(x) \leq -ax + 1/C$. We can take (see Figure 2)

$$k(x) := -(2h+1)\frac{1}{C} \quad \text{for} \quad \frac{2}{aC}h < x \leq (h+1)\frac{2}{aC}. \tag{3.4}$$

This controller drives any state belonging to I into J in one step. In this case

$$N(C) = 2\left\lceil \frac{|a|}{2}C \right\rceil.$$

Notice that

$$T_m(C) = \sum_{n=1}^{\infty} \lambda_I[T_J \geq n] = \lambda_I[T_J \geq 1] = 1 - \lambda_I[J] = 1 - 1/C$$

which converges to 1.

The previous strategy can be extended to a more general class of stabilizing quantized feedback strategies. Fix $\tau \in \mathbb{N}$ and consider the intervals

$$I_k := [-C^{\frac{k-\tau}{\tau}}, C^{\frac{k-\tau}{\tau}}], \quad k = 0, 1, \dots, \tau.$$

Notice that $I_0 = J$ and $I_\tau = I$. Consider any quantized feedback map k such that the closed loop map Γ is such that $\Gamma(I_{k+1}) \subseteq I_k$. This happens if and only if

$$|\Gamma(x)| \leq C^{\frac{k-\tau}{\tau}} \qquad \forall x \in I_{k+1} \setminus I_k.$$

This can be satisfied using a quantized feedback map k with

$$N(C) = 2\sum_{k=2}^{\tau} \lceil |a|(C^{\frac{1}{\tau}} - 1)\rceil + \lceil |a| C^{\frac{1}{\tau}}\rceil = 2(\tau-1)\lceil |a|(C^{\frac{1}{\tau}} - 1)\rceil + \lceil |a| C^{\frac{1}{\tau}}\rceil \simeq (2\tau - 1)\lceil |a| C^{\frac{1}{\tau}}\rceil.$$

In this way we have constructed an τ-steps deadbeat quantized feedback. In order to compute the expected convergence time we need the following lemma which can be easile proved.

Lemma 3.1. *Let I be an interval and $\Gamma : I \to I$ a piecewise affine map with slope a and N continuity intervals. Then for any subinterval J of I and for all $k \in \mathbb{N}$ we have that*

$$\lambda_I[\Gamma^{-k}(J)] \leq \left(\frac{N}{|a|}\right)^k \lambda_I[J]$$

Notice that, using Lemma 3.1, we can argue that

$$\lambda_I[T_J \leq \tau - 1] = \lambda_I[\Gamma^{-\tau+1}(J)] \leq \left(\frac{N(C)}{|a|}\right)^{\tau-1} C^{-1} \longrightarrow 0$$

as $C \to \infty$, since $N(C)^{\tau-1}C^{-1}$ converges to zero as $C \to \infty$. This implies that $\lambda_I[T_J \geq \tau] \longrightarrow 1$ as $C \to \infty$ and so

$$\tau \geq T_m(C) = \sum_{n=1}^{\infty} \lambda_I[T_J \geq n] \geq \tau \lambda_I[T_J \geq \tau] \longrightarrow \tau$$

as $C \to \infty$. This shows that

$$\lim_{C \to \infty} T_m(C) = \tau.$$

In figure 1 the graph of closed loop map of a 2-steps deadbeat quantized feedback is presented.

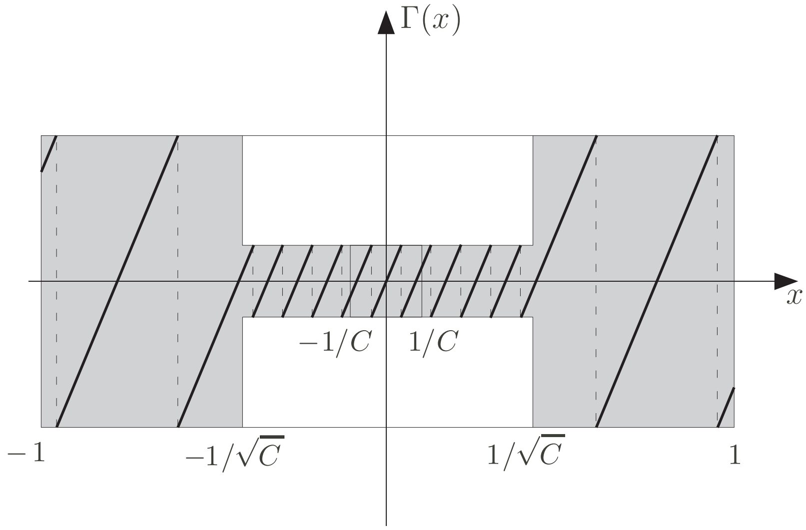

Figure 1: Graphs of the closed loop map corresponding to a 2-steps deadbeat quantized feedback.

3.2 Logarithmic quantized feedback strategy

Let $\delta \in \mathbb{R}$ such that $0 < \delta < 1$. If we impose that

$$|x_{t+1}| \leq \delta |x_t|$$

when $x_t \in I \setminus J$, then we get

$$|x_t| \leq \delta^t |x_0| \leq \delta^t$$

which implies that the state sequence converges to the interval $[-1/C, 1/C]$. Therefore a solution to the problem can be obtained by finding a feedback map $k(x)$ such that the closed loop map satisfies

$$\begin{array}{ll} |\Gamma(x)| \leq \delta |x| & \text{if} \quad 1/C \leq |x| \leq 1, \\ |\Gamma(x)| \leq 1/C & \text{if} \quad |x| \leq 1/C, \end{array}$$

and so such that

$$\begin{array}{ll} (-a-\delta)x \leq k(x) \leq (-a+\delta)x & \text{if} \quad 1/C \leq |x| \leq 1, \\ -ax - 1/C \leq k(x) \leq -ax + 1/C & \text{if} \quad |x| \leq 1/C. \end{array}$$

This means that the graph of $k(x)$ must be included in the shaded region shown in figure 2. As shown in [5] in this case the number of quantization intervals is

$$N(C) = 2\left\lceil \frac{\log C}{\log(a-\delta) - \log(a+\delta)} \right\rceil + \lceil |a| \rceil, \tag{3.5}$$

which grows logarithmically in C. For this reason this feedback map is called logarithmic quantized. Since $|x_{t-1}| > 1/C$ implies $|x_t| \leq \delta^t$, we can argue that

$$\mathbf{E}(T_J) \leq \frac{\log C}{\log(\delta^{-1})}.$$

As shown in [5], the case $\delta = 1$ yields almost stability.

3.3 Chaotic quantized feedback strategy

In [5] another possible quantized state feedback yielding convergence for almost all initial conditions has been proposed. This control strategy exploits the chaotic behavior of the state evolution inside I produced by the feedback map

$$\tilde{k}(x) := -(2h+1) \quad \text{for} \quad \frac{2}{a}h < x \leq (h+1)\frac{2}{a}, \tag{3.6}$$

assuming that $|a| > 2$. In this way we have that for almost every initial condition the state trajectory x_t is mantained inside the interval I and is dense in this interval. For this reason x_t will visit the interval J. Therefore, if we modify this feedback map in J as follows

$$k(x) = \begin{cases} \tilde{k}(x) & \text{if } x \notin J \\ -1/C & \text{if } 0 \leq x \leq 1/C \\ 1/C & \text{if } -1/C \leq x < 0. \end{cases} \tag{3.7}$$

we obtain that the state will move chaotically inside I till it will enter the interval J and there it will be entrapped. In this way we obtain a feedback map requiring

$$N(C) = 2\lceil a \rceil$$

quantization intervals and yielding this weak version of practical stability. The closed loop map $\Gamma(x)$ is shown in figure 2. In this case the evaluation of the expected convergence time will given in the sequel.

These three stabilization methods suggest that looking for a stabilizing quantized feedback with minimal quantization intervals is rather naive. In fact the last strategy would be clearly the optimal one. This is not true since the different strategies requires different information flow, but they provides closed loop systems with different stability performances.

4 Optimality of the deadbeat quantized feedback

As a first application of the previous definitions we will show now that the τ-step deadbeat quantized feedback strategy $\mathbb{K}$ presented above is optimal. Let $\mathbb{K}'$ be a stabilizing quantized feedback strategy such that

$$\limsup_{C\to\infty} \frac{N'(C)}{N(C)} \leq 1.$$

Notice that by Lemma 3.1

$$\lambda_I[T'_J \leq \tau - 1] \leq \left(\frac{N'(C)}{|a|}\right)^{\tau-1} C^{-1}$$

and so

$$\limsup_{C\to\infty} \lambda_I[T'_J \leq \tau - 1] \leq \limsup_{C\to\infty} \frac{N'(C)^{\tau-1}}{N(C)^{\tau-1}} \limsup_{C\to\ infty} \frac{N(C)^{\tau-1}}{|a|^{\tau-1}C} = 0$$

since $N(C)^{\tau-1}/C$ converges to zero. This implies that

$$\liminf_{C\to\infty} \lambda_I[T'_J \geq \tau] = 1.$$

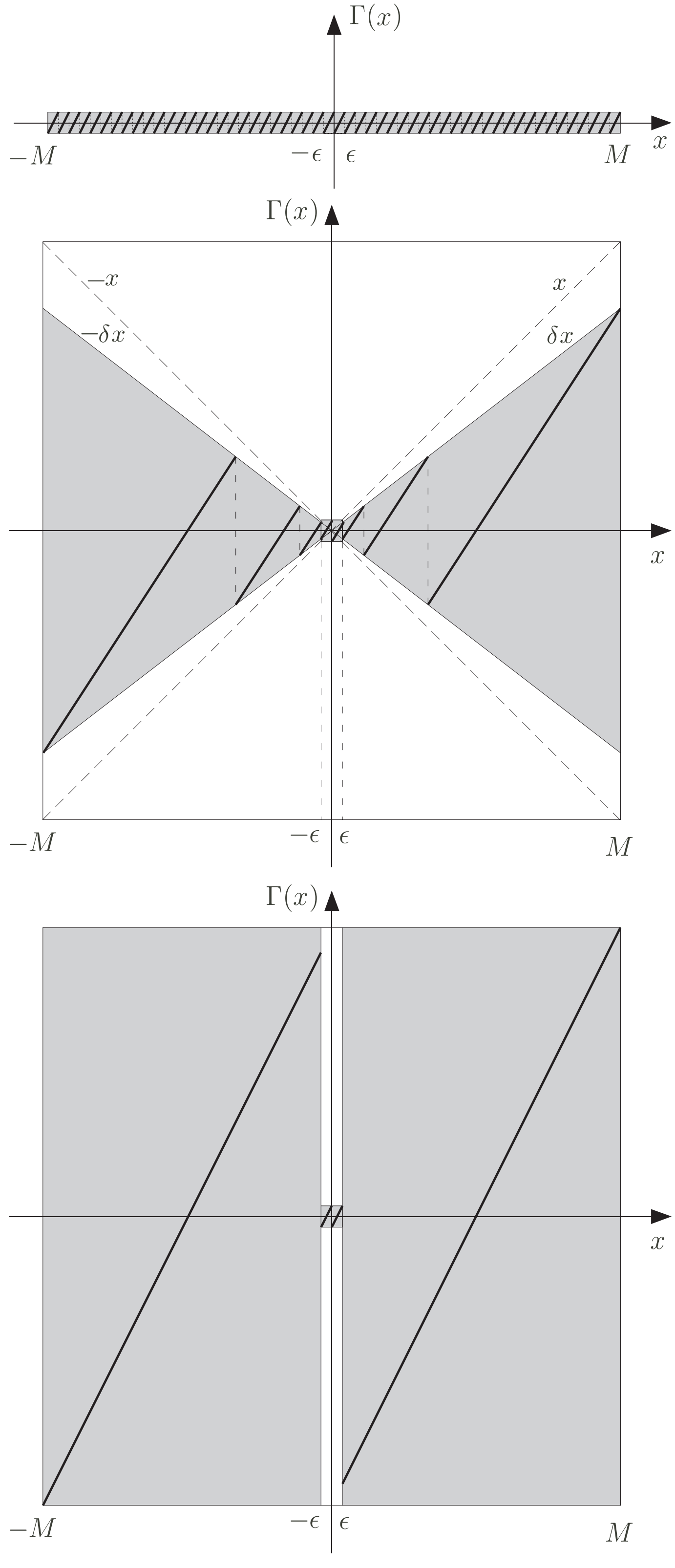

Figure 2: Graphs of the closed loop maps of the three different quantized feedback strategies ($a = 2$).

Notice moreover that

$$\liminf_{C\to\infty} T'_m(C) \geq \liminf_{C\to\infty} \tau\lambda_I[T'_J \geq \tau] = \tau$$

which implies that

$$\liminf_{C\to\infty} \frac{T'_m(C)}{T_m(C)} = 1$$

and so one of the two optimality conditions are proved. Assume conversely that

$$\limsup_{C\to\infty} \frac{T'_m(C)}{T_m(C)} \leq 1.$$

Notice that $T'_m(C) \geq (\tau+1)\lambda_I[T'_J \geq \tau+1]$ and so, using Lemma 3.1, we can argue that

$$1 - \frac{T'_m(C)}{\tau+1} \leq \lambda_I[T'_J \leq \tau] \leq \frac{N'(C)^\tau}{|a|^\tau C}.$$

This implies that

$$\begin{aligned}\limsup_{C\to\infty} \frac{N'(C)}{N(C)} &= \limsup_{C\to\infty} \frac{N'(C)}{|a|C^{1/\tau}} \limsup_{C\to\infty} \frac{|a|C^{1/\tau}}{N(C)} \geq \\ &\geq \frac{1}{2\tau-1} \limsup_{C\to\infty} \left(1 - \frac{T'_m(C)}{\tau+1}\right)^{1/\tau} \geq \frac{1}{(2\tau-1)(\tau+1)^{1/\tau}}\end{aligned}$$

which proves the other of the two optimality conditions.

5 Symbolic dynamics for expanding piecewise affine maps

In order to analyze the chaotic and logarithmic schemes, we now introduce the symbolic dynamics formalism (see [2] for more details), a very powerful technique to study dynamical systems as piecewise affine maps. In this chapter we explicitely assume that a is such that $|a| > 2$. Most of the result we present can actually be extended to the more general situation $|a| > 1$ but the technical complication is quite heavy.

Let $\Gamma : I \to I$ be a piecewise affine map of the type introduced above. We can write

$$I = I_1 \cup I_2 \cup \cdots \cup I_N,$$

where the I_h's are disjoint intervals on which Γ is affine with fixed slope a (with $|a| > 2$). Define the finite set

$$\mathcal{I} = \{I_1, I_2, \ldots, I_N\}.$$

With the map Γ we can now associate a shift over the finite alphabet $\mathcal{I}$ in the following way. Define a map $\psi : I \to \mathcal{I}^{\mathbb{N}}$ by

$$\psi(x)_n = \omega_n \ \text{ if } \ \Gamma^n(x) \in \omega_n$$

$\psi(x)$ is said to be the *code sequence* of x. If we denote by σ the left shift on $\mathcal{I}^{\mathbb{N}}$ we have that $\psi \circ \Gamma = \sigma \circ \psi$ on I. Moreover, since Γ is locally expanding, it can be shown that ψ is injective. Hence, in this case, ψ conjugates the action of Γ on I to the action of σ on the set of code sequences

$\psi(I)$. This set is in general not closed in $\mathcal{I}^{\mathbb{N}}$ for the usual direct product topology and it is useful to consider its closure

$$\Sigma_+(\Gamma) = \overline{\psi(I)}$$

which is called the *subshift associated with* the pair (Γ, I). All the dynamical and statistical properties of the map Γ can in principle be read out of the subshift $\Sigma_+(\Gamma)$; in particular this is true for the calculation of the mean entrance time for the chaotic quantized feedback. To see this we need to better exploit the symbolic structure we have introduced. Consider the *language* $\Sigma^*(\Gamma)$ associated with $\Sigma_+(\Gamma)$, namely, the set of finite words appearing in the infinite sequences in $\Sigma_+(\Gamma)$. This implies that

$$\omega_0\omega_1\cdots\omega_n \in \Sigma_*(\Gamma) \text{ if and only if } \omega_0 \cap \Gamma^{-1}\omega_1 \cap \cdots \cap \Gamma^{-n}\omega_n \neq \emptyset.$$

Consider now the directed graph with set of vertices $V = \Sigma_*(\Gamma)$ and set of edges $\mathcal{E}$ given by

$$(\omega_0\omega_1\cdots\omega_n \to \omega_0\omega_1\cdots\omega_n\omega_{n+1}) \in \mathcal{E} \qquad \text{if and only if } \omega_0\omega_1\cdots\omega_n\omega_{n+1} \in \Sigma_*(\Gamma). \tag{5.8}$$

Consider now the following labelling ξ on the edges $\mathcal{E}$:

$$\xi(\omega_0\omega_1\cdots\omega_n \to \omega_0\omega_1\cdots\omega_n\omega_{n+1}) = \omega_{n+1}.$$

If now we consider the labelled sequences associated to the infinite paths on the graph starting from the empty word ϵ, we exactly obtain all the sequences in $\Sigma_+(\Gamma)$. We have thus obtained a Markov representation of our subshift. This is not a very good representation since the underlying graph will not have any type of recursive structure, independently of the eventual mixing properties of $\Sigma_+(\Gamma)$. To obtain a more significant and useful representation it is sufficient to introduce an equivalence relation on vertices. To each finite word $\omega_0\omega_1\cdots\omega_n \in \Sigma_*(\Gamma)$, we associate its *symbolic future*:

$$\mathrm{fut}_\Sigma(\omega_0\omega_1\cdots\omega_n) = \{\overline{\omega} \in \Sigma_+(\Gamma) \mid \overline{\omega}_0 = \omega_n,\ \ \omega_0\omega_1\cdots\omega_n\overline{\omega}_1\overline{\omega}_2\cdots \in \Sigma_+(\Gamma)\}\,.$$

Consider also the *geometric future*:

$$\mathrm{fut}(\omega_0\omega_1\cdots\omega_n) = \Gamma^n(\omega_0 \cap \Gamma^{-1}\omega_1 \cap \cdots \cap \Gamma^{-n}\omega_n)\,.$$

It is easy to see that two words have the same symbolic future if and only if they have the same geometric future. Now define V_Γ to be the quotient of the set $V = \Sigma_*(\Gamma)$ by the equivalence relation:

$$\omega'_0\cdots\omega'_n \equiv \omega''_0\cdots\omega''_m \ \Leftrightarrow\ \mathrm{fut}_\Sigma(\omega'_0\cdots\omega'_n) = \mathrm{fut}_\Sigma(\omega'_0\cdots\omega'_m). \tag{5.9}$$

Vertices representable by words of length 1 will be called *principal vertices*. Edges and labels can be naturally redefined on $V = \Sigma_*(\Gamma)$ to obtain a new labeled graph X_Γ with still the property that the labeled sequences associated to the infinite paths on the graph X_Γ starting from empty word, correspond to all the possible sequences in $\Sigma_+(\Gamma)$.

6 Calculation of expected times

It can be shown that the the stochastic process on I given by $\{\Gamma^n \lambda_I \,|\, n \in \mathbb{N}\}$ induces a natural Markov chain structure on the above graph X_Γ. Using typical techniques for the calculations of first entrance time of a Markov chain process into a particular target state we can prove the following result.

Theorem 6.1. *Assume that the piecewise affine map Γ is such that the associated graph X_Γ is covering (for any vertex $x_0 \in V_\Gamma$ there exists $n \in \mathbb{N}$ such that there are paths of length n in the graph connecting x_0 to any principal vertex), then there exist positive constants r and s such that for any subinterval $J \subseteq I$ of the type*

$$J = \omega_0 \cap \Gamma^{-1}\omega_1 \cap \cdots \cap \Gamma^{-n}\omega_n,$$

where $\omega_0, \ldots, \omega_n \in \mathcal{I}$, we have that

$$\mathbb{E}(\tilde{T}_J) \leq rC + s,$$

where $\tilde{T}_J$ denotes the first entrance time into J.

It turns out that the the piecewise affine map $\tilde{\Gamma}$ associated with the quantized feedback $\tilde{k}$ defined in (3.6) satisfies the properties of the above theorem. We are now ready to give an estimation of the mean entrance time of the quantized chaotic scheme. We have the following

Corollary 6.1. *Let Γ be the piecewise affine maps introduced in (3.7) where $J \subseteq I$ is of the type*

$$J = \omega_0 \cap \Gamma^{-1}\omega_1 \cap \cdots \cap \Gamma^{-n}\omega_n.$$

Then,

$$\mathbb{E}(T_J) \leq rC + s$$

Proof If we denote, as in the previous theorem, by $\tilde{T}_J$ the first entrance time into J of the map $\tilde{\Gamma}$, we clearly have, by the way Γ is defined, that

$$\mathbb{E}(T_J) = \mathbb{E}(\tilde{T}_J) \leq rC + s\,.$$

■

Remark We believe that the applications of the Markov chain structure goes much beyond the results we just presented. Indeed we believe that this formalism should allow to obtain sharper mean time estimations in particular cases and also to analyze questions of approximation of the closed loop behavior with finite state Markov chain.

7 Estimation and optimality results

We now want to obtain lower bounds on the mean entrance time for general almost-stable piecewise affine maps. To do this we need a slight modification of the symbolic formalism.

Let $\Gamma : I \to I$ be a piecewise affine map of the type introduced above and let $J \subseteq I$ be another invariant interval. We can write

$$J = J_1 \cup J_2 \cup \cdots \cup J_M, \qquad I = I_1 \cup I_2 \cup \cdots \cup I_N \cup J,$$

where the I_h's and the J_l's are intervals on which Γ is affine with fixed slope a (such that $|a| > 2$). Define the finite sets

$$\mathcal{I} = \{I_1, I_2, \dots I_N\}, \quad \mathcal{J} = \{J_1, J_2, \dots J_M\}.$$

With the map Γ we can now associate the shift $\Sigma_+(\Gamma)$ over the finite alphabet $\mathcal{I} \cup \mathcal{J}$ as done above. We will be particularly interested in the *language* $\Sigma^*(\Gamma)$ and in the sublanguage $\Sigma^*(\Gamma, \mathcal{I}) = \Sigma_*(\Gamma) \cap \mathcal{I}^*$. We denote by $\eta_{k,h}$ the number of distinct words in $\Sigma^*(\Gamma, \mathcal{I})$, of length equal to k, and starting with the symbol $\omega_0 = I_h$. The following result shows another application of the symbolic formalism to the problem of the computation of the expected convergence time.

Lemma 7.1. *Given any $n \in \mathbb{N}$ we have that:*

$$\lambda_I(T_J \geq n) \geq \sum_{h=1}^{N} \left[\lambda_I(I_h) - \sum_{k=1}^{n-1} \frac{\lambda_I(J)}{|a|^k} \eta_{k,h} \right] \tag{7.10}$$

Proof Trivial for $n = 1$. Assume it to hold for n and let us prove it for $n + 1$.

Notice first of all that the subintervals

$$\omega_0 \cap \Gamma^{-1}(\omega_1) \cap \cdots \cap \Gamma^{-(n-1)}(\omega_{n-1}) \cap \Gamma^{-n} J_l$$

as $\omega_0, \dots, \omega_{n-1}$ vary in $\mathcal{I}$ and J_l varies in $\mathcal{J}$, form a disjoint family of intervals whose union coincide with the points of I which end inside J in exactly n steps. Moreover, since Γ^n is affine on each of these intervals it follows that

$$\lambda_I(\omega_0 \cap \Gamma^{-1}(\omega_1) \cap \cdots \cap \Gamma^{-(n-1)}(\omega_{n-1}) \cap \Gamma^{-n} J_l) \leq \frac{\lambda_I(J)}{|a|^n}.$$

Denote now by $\tilde{\eta}_{n+1,h}$ the number of distinct words on $\Sigma^*(\Gamma)$, of length equal to $n + 1$, of the type

$$I_h \omega_1 \cdots \omega_{n-1} J_l,$$

where $\omega_1, \dots, \omega_{n-1} \in \mathcal{I}$ and $J_l \in \mathcal{J}$. We clearly have that

$$\tilde{\eta}_{n+1,h} \leq \eta_{n,h} \qquad \forall n \geq 0 \;\; \forall h = 1, \dots, N\,.$$

From all previous considerations it now follows that

$$\lambda_I(T_J = n) \leq \sum_{h=1}^{N} \frac{\lambda_I(J)}{|a|^n} \eta_{n,h}.$$

We can now write

$$\begin{aligned}
\lambda_I(T_J \geq n+1) &= \lambda_I(T_J \geq n) - \lambda_I(T_J = n) \\
&\geq \lambda_I(T_J \geq n) - \sum_{h=1}^{N} \frac{\lambda_I(J)}{|a|^n} \eta_{n,h} \\
&\geq \sum_{h=1}^{N} \left[\lambda_I(I_h) - \sum_{k=1}^{n-1} \frac{\lambda_I(J)}{|a|^k} \eta_{k,h} - \sum_{h=1}^{N} \frac{\lambda_I(J)}{|a|^n} \eta_{n,h} \right] \\
&= \sum_{h=1}^{N} \left[\lambda_I(I_h) - \sum_{k=1}^{n} \frac{\lambda_I(J)}{|a|^k} \eta_{k,h} \right]
\end{aligned}$$

$\blacksquare$

Remark Formula (7.10) established in the previous lemma can lead to lower bounds on the mean entrance time, once we have upper bounds on the numbers $\eta_{k,h}$. Standard considerations on piecewise affine maps show that the entropy of the dynamical system Γ is $\log |a|$; for the conjugacy this must also be the entropy of the subshift $\Sigma^+(\Gamma)$. For the way entropy is defined for subshifts it follows that, we must have an inequality of the type

$$\eta_{k,h} \leq r(|a| + \epsilon)^k$$

where $\epsilon > 0$ can be chosen arbitrarily, while $r > 0$ depends on the particular shift $\Sigma^+(\Gamma)$ and also on ϵ. This estimation turns out to be of very little use in our investigations, since their exponential growth easily implies, if used in (7.10), that the right hand term becomes too soon negative. Moreover, the fact that the constant r not only depend on a but, in principle, on the whole structure of Γ make things unuseful for the type of general bounds we are looking for. The particular structure of the subshift we have in our case allows however a more refined analysis. A lengthy and involved computation shows the following lemma.

Lemma 7.2. *There exist constants M_1 and M_2 (only depending on the slope a) such that, for any $h = 1, \ldots, N$ we have*

$$\eta_{k,h} \leq M_1 |a|^k \sum_{t=1}^{\min\{k,N\}} \binom{N}{t} \binom{k-1}{t-1} \left(\frac{N^2 M_2}{t^2} \right)^t$$

The two previous lemmas easily lead to the following

Proposition 7.1. *For every $\overline{n} \in \mathbb{N}$ the following estimation holds*

$$\mathbb{E}(T_J) \geq \overline{n}(1 - C^{-1}) - M_1 C^{-1} (\overline{n} + 1) \binom{N + \overline{n} - 1}{\overline{n} - 1} \max \left\{ \left[\frac{N^2 M_2}{t^2} \right]^t | 1 \leq t \leq \min\{\overline{n}, N\} \right\} \tag{7.11}$$

Dependently on the growth assumed on N as a function of C, we can choose $\overline{n}$ appropriately and thus obtain upper bound estimations. Instances of the type of results we can obtain are the following two corollaries.

Corollary 7.1. *There exists positive constants r, s such that*

$$N \leq r \ln C \ \Rightarrow \ \mathbb{E}(T_J) \geq s N C^{1/N}$$

Proof Direct computations show that

$$\left[\frac{N^2 M_2}{t^2} \right]^t \leq e^{\frac{N\sqrt{M_2}}{e}},$$

$$\binom{N + \overline{n} - 1}{\overline{n} - 1} \leq M_3 e^N \left(1 + \frac{\overline{n}}{N} \right)^N .$$

Using this estimations inside (7.11) we obtain

$$\mathbb{E}(T_J) \geq \overline{n}\left[1 - C^{-1} - \frac{M_1 M_3}{N} C^{-1} e^{\left(1+\frac{\sqrt{M_2}}{e}\right)N}\left(1+\frac{\overline{n}}{N}\right)^N\right] - M_1 M_3 C^{-1} e^{\left(1+\frac{\sqrt{M_2}}{e}\right)N}\left(1+\frac{\overline{n}}{N}\right)^N.$$

It can now easily be shown that, if $N/\ln C$ is sufficiently small, there exists a constant b only depending on M_1, M_2 and M_3, such that, if we choose

$$\overline{n} = \left\lfloor bNC^{1/N}\right\rfloor,$$

then

$$M_1 M_3 C^{-1} e^{\left(1+\frac{\sqrt{M_2}}{e}\right)N}\left(1+\frac{\overline{n}}{N}\right)^N < \frac{1}{2}.$$

A simple computation then shows the thesis. ∎

Through the same kind of arguments contained in the previous proof it is possible to show the following result.

Corollary 7.2. *For any $r > 0$, there exists $s > 0$ such that,*

$$N \leq r \ln C \;\Rightarrow\; \mathbb{E}(T_J) \geq s \ln C$$

7.1 Optimality of the logarithmic quantized feedback

Let $\mathbb{K}$ be the logarithmic quantized feedback strategy. Notice first of all that, because of Corollary 7.2 we have that there exist positive constants r_1 and r_2 such that

$$r_1 \ln C \leq T_m(C) \leq r_2 \ln C.$$

Consider now another stabilizing quantized feedback strategy $\mathbb{K}'$ such that

$$\limsup_{C\to\infty} \frac{N'(C)}{N(C)} \leq 1.$$

It then follows that $N'(C) \leq r \ln C$ for a suitable positive constant r and for C sufficiently large. It then follows from Corollary 7.2 that there exists $s > 0$ such that $T'_m(C) \geq s \ln C$ and hence we also have $T'_m(C) \geq \tilde{s} T_m(C)$ for some positive constant $\tilde{s}$ and C sufficiently large. On the other hand, assume that

$$\limsup_{C\to\infty} \frac{T'_m(C)}{T_m(C)} \leq 1.$$

and, by contradiction, assume that there exists a sequence $C_n \to +\infty$ such that

$$\lim_{n\to+\infty} \frac{N'(C_n)}{N(C_n)} = 0$$

which yields

$$\lim_{n\to+\infty} \frac{N'(C_n)}{\ln C_n} = 0.$$

Using now Corollary 7.1, if n is sufficiently large, we obtain that

$$T'(C_n) \geq sN'(C_n) C_n^{1/N'(C_n)}.$$

Write

$$N'(C_n) = \alpha_n \ln C_n$$

where α_n is infinitesimal. Then, substituting above, we obtain

$$T'_m(C_n) \geq s \ln C_n \alpha_n e^{1/\alpha_n}$$

and hence

$$\frac{T'_m(C_n)}{T_m(C_n)} \geq s\alpha_n e^{1/\alpha_n}$$

which is absurd.

7.2 Optimality of the chaotic scheme

We end with some considerations on the chaotic case. Corollary 7.1 implies that for any strategy $\mathbb{K}'$ with a fixed finite number of levels N', expected times are bounded by below by

$$T'_m(C) \geq sN'C^{1/N'}$$

This is clearly not sufficient to prove the optimality of the chaotic scheme. A more refined analysis actually allows, in this case, to strengthen the above estimation as

$$T'_m(C) \geq sN'C$$

This would still not allow to conclude the optimality of the chaotic scheme because the estimation result in Corollary 6.1 is only for a particular class of target intervals J's, but it certainly says that we can not hope to find a 'better' strategy with the same number of quantization levels. These questions will be discussed elsewhere.

Acknowledgement: Research supported in part by Grants from the AFOSR and the NSF.

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