On the Sensitivity of Algebraic Riccati Equations

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Abstract

Consider the continuous-time algebraic Riccati equation (CARE) and the discrete-time algebraic Riccati equation (DARE) which arise in linear control and system theory. Appropriate assumptions on the coefficient matrices guarantee the existence and uniqueness of symmetric positive semidefinite stabilizing solutions. In this paper, we apply the theory of condition developed by Rice to define condition numbers of the CARE and DARE in the Frobenius norm, and derive explicit expressions of the condition numbers in a uniform manner.

1 Introduction

Consider the continuous-time algebraic Riccati equation (CARE)

$$Q + A^T X + XA - XBR^{-1}B^T X = 0, \tag{1.1}$$

and the discrete-time algebraic Riccati equation (DARE)

$$X - A^T XA + A^T XB(R + B^T XB)^{-1}B^T XA - C^T C = 0. \tag{1.2}$$

Appropriate assumptions on the coefficient matrices guarantee the existence and uniqueness of symmetric positive semidefinite (p.s.d.) stabilizing solutions. The equations (1.1) and (1.2) arise naturally in linear control and system theory, and there are many contributions in the literature on the theory, applications, and numerical solution of the equations.

Condition numbers of the CARE and DARE, as measures of the sensitivity of the symmetric p.s.d. stabilizing solutions to small changes in the coefficient matrices, play a key role in the perturbation theory for the CARE and DARE.

Condition numbers of the CARE and DARE have been studied by many authors (See, e.g., [1]-[4], [6]). However, there was no a uniform treatment with reasonable definitions and with complete and rigorous proofs in the literature. In this paper, we apply the theory of condition developed by Rice [5] to define condition numbers of the CARE and DARE, and derive explicit expressions of the condition numbers in a uniform manner.

In the following we take example by the CARE.

The coefficient matrices of the CARE (1.1) are $A \in \mathcal{R}^{n\times n}$, $B \in \mathcal{R}^{n\times m}$, $Q \in \mathcal{S}^{n\times n}$ (the set of $n \times n$ real symmetric matrices), and $R \in \mathcal{S}^{m\times m}$, in which $Q \geq 0$ and $R > 0$. Let $G = BR^{-1}B^T$. Then the CARE (1.1) can be written in the simplified form

$$Q + A^T X + XA - XGX = 0, \tag{1.3}$$

where $Q, G \geq 0$.

We assume that (A, G) is a stabilizable pair, and (A, Q) is a detectable pair. It is known that in such a case there exists a unique symmetric p.s.d. solution X to the CARE (1.3), and the matrix $A - GX$ is stable.

2 Definition

Consider the CARE (1.3), where $Q, G \geq 0$, (A, G) is stabilizable, and (A, Q) is detectable. Let X be the unique symmetric p.s.d. stabilizing solution to the CARE (1.3), and let Q, A and G be slightly perturbed to $\tilde{Q}$, $\tilde{A}$ and $\tilde{G}$, respectively, where

$$\begin{array}{lll} \tilde{Q} = Q + \Delta Q \in \mathcal{S}^{n\times n}, & \tilde{Q} \geq 0, & \tilde{A} = A + \Delta A \in \mathcal{R}^{n\times n}, \\ \tilde{G} = G + \Delta G \in \mathcal{S}^{n\times n}, & \tilde{G} \geq 0. & \end{array}$$

Assume that the CARE (1.3) is perturbed to the CARE

$$\tilde{Q} + \tilde{A}^T \tilde{X} + \tilde{X}\tilde{A} - \tilde{X}\tilde{G}\tilde{X} = 0. \tag{2.1}$$

It has been proved by Sun [6, Theorem 3.1] that if $\tilde{Q}, \tilde{G} \geq 0$, and if $\|(\Delta Q, \Delta A, \Delta G)\|_F$ is sufficiently small, then there is a unique symmetric p.s.d. stabilizing solution $\tilde{X}$ to the perturbed CARE (2.1), and

$$\Delta X \equiv \tilde{X} - X = \mathbf{L}^{-1}\omega(\Delta Q, \Delta A, \Delta G) + O(\|(\Delta Q, \Delta A, \Delta G)\|_F^2) \tag{2.2}$$

as $(\Delta Q, \Delta A, \Delta G) \to 0$, where

$$\omega(\Delta Q, \Delta A, \Delta G) = \Delta Q + (X\Delta A + \Delta A^T X) - X\Delta G X,$$

and the operator $\mathbf{L}$ is defined by

$$\mathbf{L}W = (A - GX)^T W + W(A - GX), \quad W \in \mathcal{S}^{n\times n}.$$

Consequently, by the theory of condition developed by Rice [5] we may define the condition number $c(X)$ by

$$c(X) = \lim_{\delta\to 0} \sup_{\substack{\left\|\left(\frac{\Delta Q}{\kappa}, \frac{\Delta A}{\alpha}, \frac{\Delta G}{\gamma}\right)\right\|_F \leq \delta \\ \Delta Q, \Delta G \in \mathcal{S}^{n\times n},\ \Delta A \in \mathcal{R}^{n\times n} \\ Q + \Delta Q \geq 0,\ G + \Delta G \geq 0}} \frac{\|\Delta X\|_F}{\xi\delta}, \tag{2.3}$$

where $\xi, \kappa, \alpha, \gamma$ are positive parameters. Taking $\xi = \kappa = \alpha = \gamma = 1$ gives the absolute condition number $c_{\text{abs}}(X)$, and taking $\xi = \|X\|_F, \kappa = \|Q\|_F, \alpha = \|A\|_F$ and $\gamma = \|G\|_F$ gives the relative condition number $c_{\text{rel}}(X)$.

The difficulty for deriving an explicit expression of $c(X)$ lies in the fact that there are the constraints

$$\Delta Q, \Delta G \in \mathcal{S}^{n\times n} \quad \text{and} \quad Q + \Delta Q \geq 0, \quad G + \Delta G \geq 0.$$

The constraints were missing by the papers on condition numbers of the CARE and DARE in the literature.

3 Explicit Expression

Substituting the expansion (2.2) into the definition (2.3) gives

$$c(X) = \max_{\substack{\left(\frac{\Delta Q}{\kappa}, \frac{\Delta A}{\alpha}, \frac{\Delta G}{\gamma}\right) \neq 0 \\ \Delta Q, \Delta G \in \mathcal{S}^{n\times n},\ \Delta A \in \mathcal{R}^{n\times n} \\ Q+\Delta Q \geq 0,\ \ G+\Delta G \geq 0}} \frac{\left\|\mathbf{L}^{-1}\omega(\Delta Q, \Delta A, \Delta G)\right\|_F}{\xi \left\|\left(\frac{\Delta Q}{\kappa}, \frac{\Delta A}{\alpha}, \frac{\Delta G}{\gamma}\right)\right\|_F}.$$

The key step for deriving an explicit expression of $c(X)$ is to prove the expression

$$c(X) = \max_{\substack{\left(\frac{\Delta Q}{\kappa}, \frac{\Delta A}{\alpha}, \frac{\Delta G}{\gamma}\right) \neq 0 \\ \Delta Q, \Delta A, \Delta G \in \mathcal{R}^{n\times n}}} \frac{\left\|\mathbf{T}^{-1}\omega(\Delta Q, \Delta A, \Delta G)\right\|_F}{\xi \left\|\left(\frac{\Delta Q}{\kappa}, \frac{\Delta A}{\alpha}, \frac{\Delta G}{\gamma}\right)\right\|_F},$$

or equivalently,

$$c(X) = \max_{\substack{(N, E, R) \neq 0 \\ N, E, R \in \mathcal{R}^{n\times n}}} \frac{\left\|\mathbf{T}^{-1}\left[\kappa N + \alpha(XE + E^T X) - \gamma XRX\right]\right\|_F}{\xi \|(N, E, R)\|_F}, \tag{3.1}$$

where the operator $\mathbf{T}$ is defined by

$$\mathbf{T}Z = (A - GX)^T Z + Z(A - GX), \quad Z \in \mathcal{R}^{n\times n}.$$

From (3.1) we get an explicit expression of $c(X)$.

Theorem 1 [7]. *Let*

$$T = I_n \otimes (A - GX)^T + (A - GX)^T \otimes I_n,$$

and let

$$Z_1 = T^{-1}, \quad Z_2 = T^{-1}\left[I \otimes X + (X \otimes I)\Pi\right], \quad Z_3 = T^{-1}(X \otimes X),$$

where Π *is the vec-permutation matrix; i.e.,*

$$\mathsf{vec}(N^T) = \Pi \mathsf{vec}(N), \quad N \in \mathcal{R}^{n\times n}.$$

Then

$$c(X) = \frac{1}{\xi}\left\|(\kappa Z_1,\ \alpha Z_2,\ \gamma Z_3)\right\|_2. \tag{3.2}$$

From (3.2) we get the absolute condition number

$$c_{\text{abs}}(X) = \|(Z_1,\ Z_2,\ Z_3)\|_2,$$

and the relative condition number

$$c_{\text{rel}}(X) = \frac{1}{\|X\|_F}\left\|(\|Q\|_F Z_1,\ \|A\|_F Z_2,\ \|G\|_F Z_3)\right\|_2.$$

Byers [1] suggests an approximate condition number $K_B(X)$ which is expressed by

$$K_B(X) = \frac{1}{\|X\|_F}\left(\|Q\|_F\|Z_1\|_2 + \|A\|_F\|Z_2\|_2 + \|G\|_F\|Z_3\|_2\right).$$

Comparing it with $c_{\text{rel}}(X)$ gives

$$c_{\text{rel}}(X) \leq K_B(X) \leq 3c_{\text{rel}}(X).$$

4 The Complex Case

Consider the CARE

$$Q + A^H X + XA - XGX = 0, \tag{4.1}$$

where $A \in \mathcal{C}^{n\times n}, Q, G \in \mathcal{H}^{n\times n}$ (the set of $n \times n$ Hermitian matrices), $Q, G \geq 0$, (A, G) is stabilizable, and (A, Q) is detectable. It is known that in such a case there is a unique Hermitian p.s.d. stabilizing solution X to the CARE (4.1).

In the manner similar to the definition (2.3) we may define the condition number $c(X)$ by

$$c(X) = \lim_{\delta\to 0} \sup_{\substack{\left\|\left(\frac{\Delta Q}{\kappa}, \frac{\Delta A}{\alpha}, \frac{\Delta G}{\gamma}\right)\right\|_F \leq \delta \\ \Delta Q, \Delta G \in \mathcal{H}^{n\times n},\ \Delta A \in \mathcal{C}^{n\times n} \\ Q+\Delta Q \geq 0,\ \ G + \Delta G \geq 0}} \frac{\|\Delta X\|_F}{\xi\delta}.$$

The following result gives an explicit expression of $c(X)$.

Theorem [7]. *Let*

$$\begin{array}{l} T = I_n \otimes (A - GX)^H + (A-GX)^T \otimes I_n, \\ T^{-1} = S + i\Sigma, \qquad T^{-1}(I\otimes X) = U_1 + i\Omega_1, \\ T^{-1}(X^T \otimes I)\Pi = U_2 + i\Omega_2, \\ T^{-1}(X\otimes X) = V + i\Theta, \qquad i = \sqrt{-1}, \end{array}$$

and let

$$Z_1^{(c)} = \begin{pmatrix} S & -\Sigma \\ \Sigma & S \end{pmatrix}, \qquad Z_2^{(c)} = \begin{pmatrix} U_1 + U_2 & \Omega_2 - \Omega_1 \\ \Omega_1 + \Omega_2 & U_1 - U_2 \end{pmatrix},$$

$$Z_3^{(c)} = \begin{pmatrix} V & -\Theta \\ \Theta & V \end{pmatrix},$$

where $S, \Sigma, U_1, \Omega_1, U_2, \Omega_2, V, \Theta \in \mathcal{R}^{n^2\times n^2}$. *Then*

$$c(X) = \frac{1}{\xi}\left\|\left(\kappa Z_1^{(c)},\ \alpha Z_2^{(c)},\ \gamma Z_3^{(c)}\right)\right\|_2.$$

5 An Numerical Example

Example 1 (Byers [1]). Consider the CARE (1.3) with

$$Q = C^T C \quad \text{with} \quad C = (10,\, 100), \qquad A = \begin{pmatrix} -0.100 & 0.000 \\ 0.000 & -0.020 \end{pmatrix},$$

$$G = BR^{-1}B^T \quad \text{with} \quad B = \begin{pmatrix} 0.100 & 0.000 \\ 0.001 & 0.010 \end{pmatrix} \quad \text{and} \quad R = \begin{pmatrix} 1 + 10^{-m} & 1 \\ 1 & 1 \end{pmatrix}.$$

The pair (A, G) is stabilizable, and the pair (A, Q) is detectable.

By using the MATLAB file “are” one can compute an approximation of the unique symmetric p.s.d. stabilizing solution X to the CARE (1.3) with a high accuracy, and then compute the condition number $c(X)$ by the formula (3.2). Some results are listed in Table 1.

Table 1 ($j = 12$)

m	$c_{\text{abs}}(X)$	$c_{\text{rel}}(X)$
0	4.88×10^7	4.98×10
1	4.91×10^7	5.03×10^2
2	5.10×10^7	5.27×10^3
3	5.54×10^7	5.85×10^4
4	5.91×10^7	6.34×10^5
5	6.08×10^7	6.57×10^6

From the results listed in Table 1 we see that the CARE of this example is ill-conditioned in the absolute sense, and it is ill-conditioned for large m in the relative sense.

6 Final Remarks

The technique we presented here has been used to study condition numbers for the DARE and the periodic DARE. Both the real case and complex case are considered.

The problem of how to develop practical algorithms for computing condition numbers by using the explicit expressions with large n is a research problem.

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