Bifurcations of the Controlled Escape Equation

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Abstract

In this paper we present numerical methods for the analysis of nonlinear autonomous control systems and two conditions, local accessibility and an inner-pair condition, under which they can be applied. These methods can be extended to work also for systems with time-periodic right hand side. In particular, the escape equation with sinusoidal driving term and additional control is analyzed. We will show that its stability behavior undergoes interesting bifurcations when the range of the control influence is varied.

1 Introduction

In this paper we consider a family of nonlinear control systems

$$\dot{x}(t) = f(x(t), u(t)) \tag{1.1}$$

on a connected smooth manifold M with dimension d where $f : M \times \mathbb{R}^m \to \mathbf{T}M$ is smooth and the controls u are taken from the set $\mathcal{U}^\rho := \{u \in L_\infty(\mathbb{R}, \mathbb{R}^m), u(t) \in \rho \cdot U$ for almost all $t \in \mathbb{R}\}$ for a non-void, convex, and compact subset U of $\mathbb{R}^m$ with $0 \in \text{int}U$ and a parameter $\rho \in [0, \rho^*]$. Relevance of the parameter dependence will be indicated by a superfix ρ. We assume for every initial value $x \in M$ and every $u \in \mathcal{U}^{\rho^*}$ that there exists a unique solution $\phi(t; x, u), t \in \mathbb{R}$, with $\phi(0; x, u) = x$. Furthermore, by restricting ourselves to a compact invariant subset of the state space we can focus on compact M only. Throughout we assume the system to be locally accessible for all $\rho > 0$, i.e. for all $x \in M$ and all $T > 0$ one has $\text{int}\mathcal{O}^{\rho,+}_{\leq T}(x) \neq \emptyset$, and $\text{int}\mathcal{O}^{\rho,-}_{\leq T}(x) \neq \emptyset$, where $\mathcal{O}^{\rho,+}_{\leq T}(x) = \{y \in M, y = \phi(t; x, u)$ with $0 \leq t \leq T$ and $u \in \mathcal{U}^\rho\}$ and $\mathcal{O}^{\rho,-}_{\leq T}(x) = \{y \in M, x = \phi(t; y, u)$ with $0 \leq t \leq T$ and $u \in \mathcal{U}^\rho\}$ are the positive and negative reachable sets from x.

The limit behavior of this system is determined by its control sets, i.e. maximal subsets of M where complete approximate controllability holds, and its chain control sets (see [1]). Under an inner-pair condition the chain control sets and the control sets generically coincide. We will see in section 2 that this inner-pair condition holds for n-th order equations with additive control and therefore under change of the parameter ρ the control sets will change continuously for all but countably many $\rho \in [0, \rho^*]$. If the right hand side of the control system has a t-periodic driving term as only explicit time dependence, the time can be interpreted as a further space dimension and control sets and chain control sets can be studied for the corresponding system without direct time dependence. We show that if in this case control sets merge under variation of the parameter ρ, then they merge over the whole time-period at once.

Only in very simple cases can control sets be found analytically. Thus numerical methods are an important tool for the analysis of control systems. In section 3 we briefly present a method for the computation of control sets that was developed by Szolnoki (see [7]) and is based on an algorithm for the computation of relative global attractors and unstable manifolds of dynamical systems by Dellnitz, Hohmann and Junge (see [3] and [4]).

In section 4 finally we will clarify our findings by numerical results for the controlled escape equation.

2 Control sets and chain control sets

Control sets are maximal subsets of the state space where complete approximate controllability holds. A control set C is called invariant if $\mathrm{cl}C = \mathrm{cl}\{\phi(t;x,u), u \in \mathcal{U}, t \geq 0\}$. If we allow the solutions to make small jumps, we are led to the notion of chain-controllability: For $\epsilon, T > 0$, and $x, y \in M$ a controlled (ϵ, T)-chain from x to y is given by $n \in \mathbb{N}$, $x_0, x_1, \ldots, x_n \in M$, $u_0, \ldots, u_{n-1} \in \mathcal{U}$, $t_0, \ldots t_{n-1} \geq T$ with $x_0 = x, x_n = y$, and

$$\mathrm{d}(\phi(t_j; x_j, u_j), x_{j+1}) \leq \epsilon \quad \text{for all } j = 0, \ldots, n-1.$$

A chain control set $E \subset M$ is a maximal subset such that (i) for all $x, y \in E$ and all $\epsilon, T > 0$ there is a controlled (ϵ, T)-chain from x to y, and (ii) for all $x \in E$ there is $u \in \mathcal{U}$ such that $\phi(t; x, u) \in E$ for all $t \in \mathbb{R}$. Every control set is contained in a chain control set.

The following inner-pair condition guarantees much closer a relationship between control sets and chain control sets (cf. [1]):

Definition 2.1. *If for all $\rho_1, \rho_2 \in [0, \rho^*]$, $\rho_1 < \rho_2$, and for all $x \in M$, $u \in \mathcal{U}^{\rho_1}$, there is $T > 0$ such that $\phi(T; x, u) \in \mathrm{int}\mathcal{O}^{+,\rho_2}(x)$, then the family of control systems (1.1) is said to fulfill the inner-pair condition.*

Theorem 2.1. *Let $E^{\rho_1} \subset M$ be a chain control set of $(1.1)^{\rho_1}$, and for each $\rho \in [\rho_1, \rho^*]$ let $E^\rho \subset M$ be the unique chain control set of $(1.1)^\rho$ with $E^{\rho_1} \subset E^\rho$. Moreover let the family (1.1) be locally accessible and fulfill the inner-pair condition.*

Then for each ρ there is precisely one control set D^ρ with $E^{\rho_1} \subset \mathrm{int}D^\rho$, and for all but countably many values of ρ the following holds:

(i) $\mathrm{cl}D^\rho = E^\rho$.

(ii) The mapping $\rho \mapsto \mathrm{cl}D^\rho$ is continuous with respect to the Hausdorff metric.

At the discontinuities control sets merge and possibly change their stability behavior.

The inner-pair condition can be extended to time dependent systems in the obvious way and the following holds.

Proposition 2.1. *Consider the family of n-th order systems*

$$y^{(n)}(t) + f(t, y, y^{(1)}, \ldots, y^{(n-1)}) = b(t, y, \ldots, y^{(n-1)})\, u(t) \tag{2.2}$$

where $f, b : \mathbb{R}^{n+1} \to \mathbb{R}$ are continuous mappings and $u \in \mathcal{U}^\rho$. If there is $\alpha > 0$ such that $|b(t, y_0, y_1, \ldots, y_{n-1})| \geq \alpha$ holds for all $(t, y_0, y_1, \ldots, y_{n-1}) \in \mathbb{R}^{n+1}$, then (2.2) fulfills the inner-pair condition.

If the right hand side in (1.1) is generalized to be explicitly periodically time dependent, the control setting can still be used. Let $f : \mathbb{R} \times M \times \mathbb{R}^m \to \mathbf{T}M$ be smooth and a system be given by $\dot{x}(t) = f(t, x(t), u(t))$. If f is T-periodic in the first coordinate: $f(t, x, u) = f(t+T, x, u)$, we can consider the associated time-independent system

$$\begin{pmatrix} \dot{x} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} f(z, x, u) \\ T/2\pi \end{pmatrix} \tag{2.3}$$

where we identify $z = 0$ and $z = 2\pi$. If control sets for such a system have contact for one $z \in \mathbb{R}/2\pi\mathbb{Z}$, they have contact for the whole interval.

Proposition 2.2. *If D_1 and D_2 are control sets for the control system (2.3) and if there is $(z_0, x_0) \in \mathbb{R}/2\pi\mathbb{Z} \times M$ such that $(z_0, x_0) \in clD_1 \cap clD_2$, then for every $z \in \mathbb{R}/2\pi\mathbb{Z}$ there is $x \in M$ such that $(z, x) \in clD_1 \cap clD_2$.*

Proof. For proofs of Propositions 2.1 and 2.2 see [5].

3 Numerical approximation of control sets

Control sets can not be approximated directly but the computation of chain control sets is almost as good according to Theorem 2.1. Szolnoki developed three algorithms whose combination is a powerful tool for locating chain control sets (see [7] and [8]). They have in common that a compact subset Q of the state space is chosen and successively divided in finer and finer partitions. A selection criterion in each step defines a decreasing sequence of subsets of Q.

A graph algorithm finds strongly connected components of a directed graph associated to the discretized dynamics. In principle these strongly connected components converge to the chain control sets if the partition gets finer but memory and time consumption do not allow for fine partitioning in higher dimensions. The subdivision algorithm approximates viability kernels $V(Q)$ relative to the start set Q. If Q contains precisely one chain control set E, then $E = V(Q)$. If Q possibly contains more than one chain control set, then in some cases a continuation algorithm can be used to create a covering of precisely one chain control set, which allows the subdivision algorithm to be used. A combination of these methods together with smart selection of the set Q often produces good results. But for systems that are sensitive to small changes in their initial conditions or for high dimensions resources are limited.

In the special case of time-periodic systems (2.3) one can spare one dimension by investigating Poincaré-cuts and using Proposition 2.2 to draw conclusions for the full dimensional system.

4 The controlled escape equation

As an example we consider the controlled escape equation

$$\begin{aligned} \ddot{x}(t) + \gamma\dot{x}(t) + x(t) - x(t)^2 &= F\sin(\omega t) + u(t) \\ \gamma, \omega, F > 0, \qquad & t \in \mathbb{R}. \end{aligned}$$

This equation has been intensively studied for instance by Soliman and Thompson (see [6]). Interpreting the controls u as bounded noise terms it models the rocking movement of a ship on sea under the influence of periodic waves and some additional disturbance. According to Proposition 2.1 the inner-pair condition is satisfied. Transferring it into the form of system (1.1) leads to

$$\begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \\ \dot{z}(t) \end{pmatrix} = \begin{pmatrix} y(t) \\ -\gamma\, y(t) - x(t) - x(t)^2 \\ \omega \end{pmatrix} + \begin{pmatrix} 0 \\ F\, \sin(z(t)) + u(t) \\ 0 \end{pmatrix},$$

where $M = \mathbb{R} \times \mathbb{R} \times \mathbb{R}/2\pi\mathbb{Z}$, $u(t) \in U^\rho := [-\rho, \rho]$, $0 \leq \rho \leq \rho^*$. We choose the parameter values $\omega = 0.85$, $\gamma = 0.1$, $F = 0.06$. The system proves too complicated for a direct three dimensional investigation. Therefore we look at four Poincaré-cuts, at $z = 0$, $z = 0.25 * 2\pi$, $z = 0.5 * 2\pi$, and $z = 0.75 * 2\pi$ and investigate the discrete dynamics induced by the time-2π-map. For the uncontrolled case $\rho = 0$ we find four fixed points. At $z_0 = 0$ FP1 is located close to $(0; 0.25)$ and is stable, FP2 is near $(-0.2; 0.4)$ and unstable, stable FP3 is close to $(-0.3; -0.4)$ and unstable FP4 finally is close to $(1; 0)$. Choosing small neighborhoods of these fixed points and using the continuation algorithm we also find the unstable and stable manifolds of the unstable fixed points (see Figure 1). Now if we go to $\rho = 0.005$, according to Theorem 2.1 around these fixed points develop invariant control sets $D1^\rho$ and $D3^\rho$ and variant control sets $D2^\rho$ and $D4^\rho$ for which we draw the domains of attractions as well. Finding a parameter value ρ for which there are three control sets is difficult. Only for $z = \pi/2$ the graph algorithm is

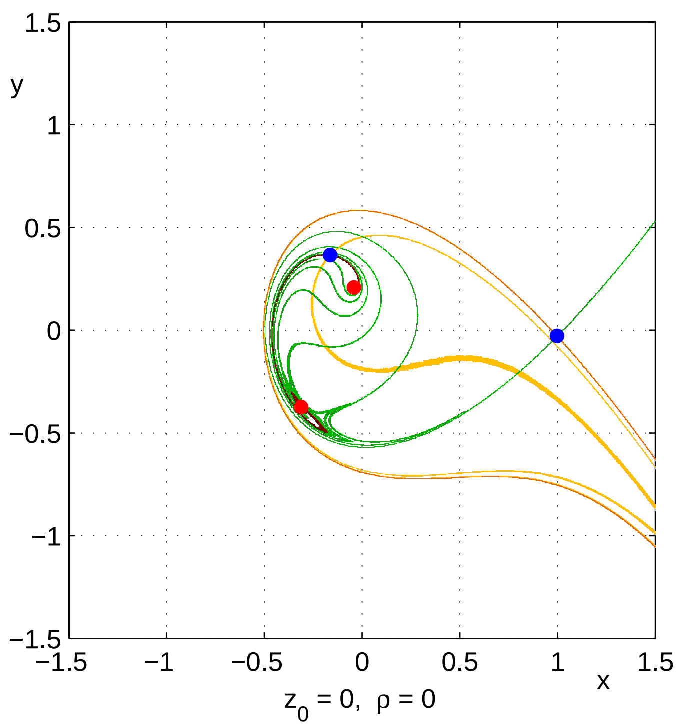

Figure 1. For $\rho = 0$ the picture shows two unstable (blue) and two stable (red) fixed points. The unstable manifold of FP4 is depicted in green, its stable one in light brown. The unstable manifold of FP2 has the yellow color, its stable one is dark brown.

able to resolve that at $\rho = 0.0085$ the control sets $D1^\rho$ and $D2^\rho$ have merged whereas $D3^\rho$ is still separated from them. But then Proposition 2.2 states that $D3^{0.0085}$ must be separate for all z and therefore we can cut out $D1^{0.005}$ and apply subdivision to the rest which gives the displayed pictures. At $\rho = 0.01$ the three control sets around the origin have merged to one invariant control set. At $\rho = 0.013$ the whole inner area has become transient since now the exit via the $D4$ control set is possible. Let $\mathcal{A}(D4^\rho)$ denote the domain of attraction of $D4^\rho$ and $\mathcal{A}(D2^\rho)$ denote the domain of attraction of the variant control set that includes $D2^{0.005}$ (see Figure 2).

We get the full 3D-pictures of the control sets by inserting the results at the selected four Poincaré-cuts, computing connecting orbits and applying subdivision afterwards (see Figure 3).

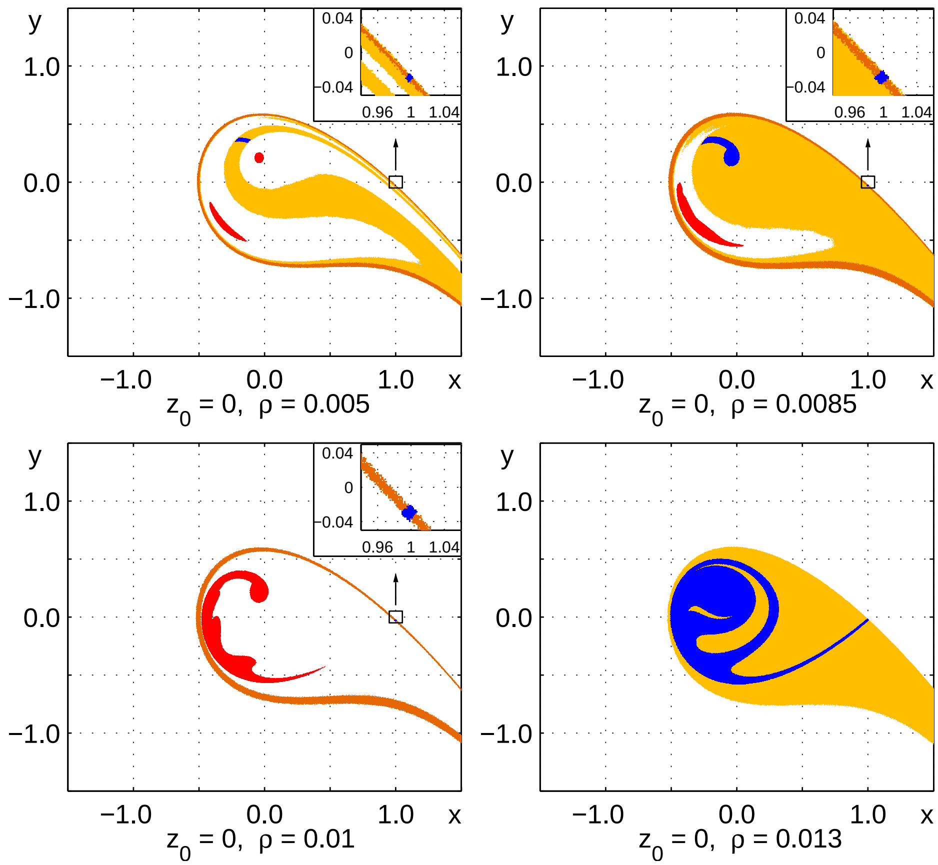

Figure 2. For $\rho = 0.005$ control sets (red = invariant, blue = variant) have developed around the fixed points. At $\rho = 0.0085$ invariant $D1^\rho$ has merged with variant $D2^\rho$ forming $D12^{0.0085}$ and lost its invariance. For $\rho = 0.01$ the sets $D12^{0.0085}$ and $D3^{0.0085}$ have merged and form one invariant control set in the middle. At $\rho = 0.013$ this control set in the middle has merged with the variant control set near $(0, 1)$ and formed one variant control set. Exit is now possible from everywhere. $\mathcal{A}(D4^\rho)$ is depicted dark brown, $\mathcal{A}(D2^\rho)$ is yellow.

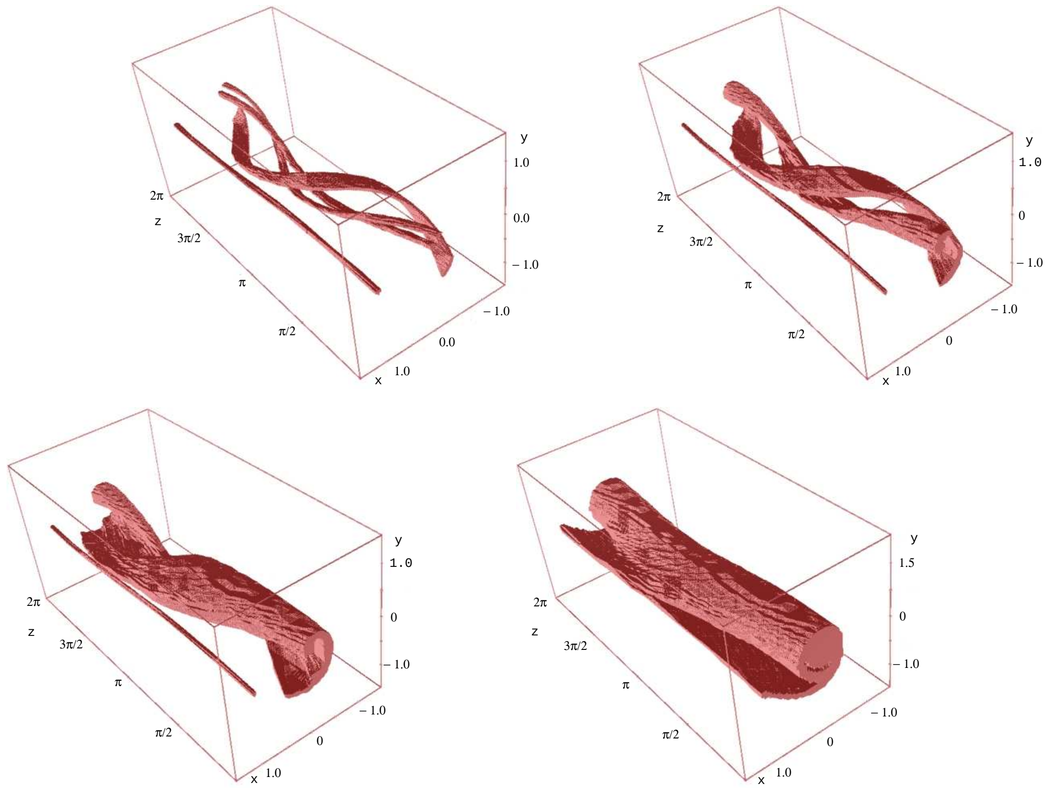

Figure 3 The 3D control sets for $\rho = 0.005, 0.0085, 0.01, 0.013$.

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