Partial Eigenvalue Assignment in Linear Systems: Existence, Uniqueness and Numerical Solution

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Abstract

The problem of reassigning a part of the open-loop spectrum of a linear system by feedback control, leaving the rest of the spectrum invariant, is called the partial eigenvalue assignment problem. In this paper, we derive new necessary and sufficient conditions for existence and uniqueness of solution of the partial eigenvalue assignment problem and then present a practical parametric algorithm to numerically solve it. The algorithm is feasible for large-scale solution and computationally viable. It also offers an opportunity to devise a robust solution to the problem by exploiting the arbitrary nature of the parameters.

1 Introduction

Given a set $S = \{\mu_1, \ldots, \mu_n\}$ of complex numbers, closed under complex conjugation, the eigenvalue assignment problem (EVA) for the linear control system

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{1.1}$$

is the problem of finding a real matrix F, called the feedback matrix, such that the spectrum of the matrix $A - BF$ is the S.

The EVA problem is one of the central problems in control system design and has been widely studied both from theoretical and computational view points. A brief account of the existing numerical methods and the conditioning of the EVA problem can be found in [4, Chapter 11].

Many practical applications such as the design of large and sparse structures, electrical networks, power systems, computer networks, etc., give rise to very large and sparse problems and the conventional numerical methods (e.g. the QR based and Schur methods) for EVA problem do not work well. Furthermore, in most of these applications only a small number of eigenvalues, which are responsible for instability and other undesirable phenomenons, need to be reassigned. Clearly, a complete eigenvalue assignment, in case when only a few eigenvalues are "bad", does not make sense.

These consideration gives rise to the following partial eigenvalue assignment problem (PEVA) for the linear control system (1.1). Given a pair (A, B) whose spectrum, $\Omega(A) = \{\lambda_1, \ldots, \lambda_p; \lambda_{p+1}, \ldots, \lambda_n\}$ and $S = \{\mu_1, \ldots, \mu_p\}$, closed closed under complex conjugation, find a real feedback matrix F such that $\Omega(A-BF) = \{\mu_1, \ldots, \mu_p; \lambda_{p+1}, \ldots, \lambda_n\}$. The PEVA problem was considered by [6] and [3] and projection algorithms were developed. In [6], conditions for existence and uniqueness for the single-input problem were given.

In this paper, we give another close-look at this problem. We prove new necessary and sufficient conditions for existence and uniqueness (nonuniqueness) for both the single-input and multi-input problems and then propose a parametric approach for solving the PEVA problem.

The major computational requirements of this new approach are solutions of a small Sylvester equation of order p and a $p \times p$ linear algebraic system. The parametric nature of the algorithm offers an opportunity to devise a numerically robust feedback matrix F.

The paper is organized as follows. In Section 2, we state the well-known criteria of controllability, and result on existence and uniqueness of solution of EVA problem. In Section 3, we state and prove our result on existence and uniqueness of solution of the PEVA problem. The new parametric approach for the multi-input PEVA problem is described in Section 4. The results of numerical experiments are displayed in Section 5.

2 Existence and Uniqueness Result for Eigenvalue Assignment Problem

In this section, we state a well known result on the existence and uniqueness of solution of the eigenvalue assignment problem. The notion of *controllability* is crucial to these results (see, for example, [4]).

Theorem 2.1. (Eigenvector Criterion of Controllability).
The system (1.1) or, equivalently, the matrix pair (A, B) is controllable *with respect to the eigenvalue λ of A if $y^H B \neq 0$ for all $y \neq 0$ such that $y^H A = \lambda y^H$.*

Definition 2.1. *The system (1.1) or the matrix pair (A, B) is* partially controllable *with respect to the subset $\{\lambda_1, \ldots, \lambda_p\}$ of the spectrum of A if it is controllable with respect to each of the eigenvalues λ_j, $j = 1, \ldots, p$.*

Definition 2.2. *The system (1.1) or the matrix pair (A, B) is* completely controllable *if it is controllable with respect to every eigenvalue of A.*

Theorem 2.2. (Existence and Uniqueness for Eigenvalue Assignment Problem). *The eigenvalue assignment problem for the pair (A, B) is solvable for any arbitrary set S if and only if (A, B) is completely controllable. The solution is unique if and only if the system is a single-input system (that is, if B is a vector). In the multi-input case, there are infinitely many solutions, whenever a solution exists.*

Proof. The proof is available in any control theory text book, *e.g.* [4, 1, 5]. □

3 Existence and Uniqueness Result for Partial Eigenvalue Assignment Problem

We now prove a result similar to Theorem 2.2 for the existence and uniqueness for the partial eigenvalue assignment problem. *This result and the proof are new.*

Theorem 3.1. (Existence and Uniqueness for Partial Eigenvalue Assignment Problem).
Let $\Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_p; \lambda_{p+1}, \ldots, \lambda_n)$ be the diagonal matrix containing the eigenvalues $\lambda_1, \ldots, \lambda_n$ of $A \in \mathbb{C}^{n\times n}$. Assume that the sets $\{\lambda_1, \ldots, \lambda_p\}$ and $\{\lambda_{p+1}, \ldots, \lambda_n\}$ are disjoint. Let the eigenvalues $\lambda_1, \ldots, \lambda_p$ to be changed to $\mu_1, \ldots, \mu_p$ and the remaining eigenvalues to stay invariant.

Then the partial eigenvalue assignment problem for the pair (A, B) is solvable for any choice of the closed-loop eigenvalues $\mu_1, \ldots, \mu_p$ if and only if the pair (A, B) is partially controllable with respect to the set $\{\lambda_1, \ldots, \lambda_p\}$. The solution is unique if and only if the system is a completely controllable single-input system. In the multi-input case, and in the single-input case when the system is not completely controllable, there are infinitely many solutions, whenever a solution exists.

Proof. We first prove the *necessity.* Suppose the pair (A, B) is not controllable with respect to some λ_j, $1 \le j \le p$. Then there exists a vector $y \neq 0$ such that $y^H(A - \lambda_j I) = 0$ and $y^H B = 0$. This means that for any F, we have $y^H(A - BF - \lambda_j I) = 0$, which implies that λ_j is an eigenvalue of $A - BF$ for every F, and thus λ_j cannot be reassigned.

Next we prove the *sufficiency.* Denote $\Lambda_1 = \operatorname{diag}(\lambda_1, \ldots, \lambda_p)$ and $\Lambda_2 = \operatorname{diag}(\lambda_{p+1}, \ldots, \lambda_n)$. Then we need to prove that there exists a feedback matrix F which assigns the eigenvalues in Λ_1 arbitrarily while keeping all the other eigenvalues unaltered.

Let $X = (x_1, \ldots, x_n)$ and $Y = (y_1, \ldots, y_n)$ be, respectively, the right and left eigenvector matrices of A, and let $Y_1 = (y_1, \ldots, y_p)$. Since $Y^H X = I$ and $Y^H A X = \operatorname{diag}(\Lambda_1, \Lambda_2)$, then the partial controllability of the matrix pair (A, B) with respect to eigenvalues in Λ_1 implies the partial controllability of the pair $(\operatorname{diag}(\Lambda_1, \Lambda_2), Y^H B)$ with respect to the same eigenvalues. Therefore, the pair $(\Lambda_1, Y_1^H B)$ is completely controllable because $\{\lambda_1, \ldots, \lambda_p\} \cap \{\lambda_{p+1}, \ldots, \lambda_n\} = \emptyset$.

By Theorem 2.2, there exists a feedback matrix Φ such that the closed-loop matrix $\Lambda_1 - Y_1^H B\Phi$ has the desired eigenvalues $\mu_1, \ldots, \mu_p$. Denote

$$F \;=\; \Phi Y_1^H . \tag{3.2}$$

Then the eigenvalues of the closed-loop matrix are exactly as required. This is seen as follows:

$$\begin{aligned}\{\mu_1, \ldots, \mu_p, \lambda_{p+1}, \ldots, \lambda_n\} = \Omega\left(\operatorname{diag}(\Lambda_1, \Lambda_2) - Y^H B(\Phi, 0)\right) = \\ \Omega\left(Y^H\left(A - B((\Phi, 0)Y^H)\right)X\right) = \Omega\left(A - B(\Phi Y_1^H)\right) .\end{aligned} \tag{3.3}$$

Uniqueness of the solution in the single-input case that is completely controllable and the existence of infinitely many solutions in the multi-input case follows directly from Theorem 2.2.

To complete the proof we need to show that infinitely many solutions to the partial eigenvalue assignment problem are possible when B is a vector (single-input case) and there exists an uncontrollable eigenvalue λ_k for some $k > p$ (that is, the associated k^{th} right eigenvector y_k is such that $y_k^H A = \lambda_k y_k^H$ and $y_k^H B = 0$).

Let F be a solution to the partial eigenvalue assignment problem. Denote the left and right eigenvectors of the closed-loop matrix $A_c = A - BF$ by Y_c and X_c. Clearly $y_k^H A_c = y_k^H(A - BF) = \lambda_k y_k$ and thus y_k is also the k^{th} column of Y_c. Let $F_\alpha = \alpha y_k^H$, where α is an arbitrary scalar. As in (3.3) we can show that the eigenvalues $\mu_1, \ldots, \mu_p$, $\lambda_{p+1}, \ldots, \lambda_{k-1}$, $\lambda_{k+1}, \ldots, \lambda_n$ of A_c remain unchanged by the application of feedback F_α. Furthermore, the eigenvalue λ_k of A_c also remains unchanged by the feedback F_α, since the pair (A_c, B) is not controllable with respect to λ_k by the *necessity* part of this theorem. Thus

$$\Omega\left(A - BF\right) = \Omega\left(A_c\right) = \Omega\left(A_c - BF_\alpha\right) = \Omega\left(A - B(F + \alpha y_k^H)\right),$$

showing that if F is a solution, so is $F + \alpha y_k^H$ for an arbitrary α. □

4 A Parameterization Approach for the Partial Eigenvalue Assignment

In this section, we develop a parametric approach to the partial eigenvalue assignment problems for both the first-order pair (A, B).

We remark that developing parametric solutions to these problems is useful in that one can then think of solving some other important variation of the problems, such as the robust partial eigenvalue assignment problem, by exploiting freedom of these parameters.

We make the following assumptions that will simplify the proofs of our theorems for the rest of the chapter. Justification for each of these assumptions is also stated.

Assumption 4.1. *The control matrix B has full rank.*

Justification: Indeed, if the $n \times m$ matrix B has rank $m_1 < m$, then it admits the economy-size QR decomposition $B = QR$, where R is an $m_1 \times m$ matrix of full rank (see [2]). Suppose that we have performed partial eigenvalue assignment with the full-rank matrix Q (instead of B) and obtained the feedback matrix K. Then $QK = BF = (QR)F$ and we recover the feedback matrix F for use with the original control matrix B, thus solving the underdetermined linear system $K = RF$ in the least-square sense.

Note that if the full-rank matrix B is close to the rank-deficient matrix; that is, if the absolute values of some diagonal entries of R are less than certain tolerance, then elimination of such "almost linearly dependent" parts of B via the economy-size QR decomposition might result in a better feedback matrix.

Example 4.1 (Rank deficient control matrix). *Consider the control matrix B and the feedback matrix F defined by*

$$B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix} \text{ and } F = \begin{pmatrix} 1 & 2 & 3 \\ -4 & -5 & -6 \end{pmatrix}.$$

The economy-size QR decomposition of B is

$$B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix} (\sqrt{3}, \sqrt{3}) = QR.$$

Therefore, $BF = B_{\text{new}} F_{\text{new}}$, where

$$B_{\text{new}} = Q = \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix}$$

is a full rank matrix and

$$F_{\text{new}} = RF = (-3\sqrt{3}, -3\sqrt{3}, -3\sqrt{3}).$$

Now suppose that in order to satisfy Assumption 4.1, the feedback control problem was solved with full-rank B_{new} instead of rank-deficient B and a feedback matrix

$$K = (1, 2, 3)$$

was obtained. To get the equivalent 2×3 feedback matrix F_{old} corresponding to the control matrix B, we then solve the linear system

$$K = (1, 2, 3) = (\sqrt{3}, \sqrt{3}) F_{\text{old}} = RF_{\text{old}}$$

giving

$$F_{\text{old}} = R^{\dagger} K = \begin{pmatrix} \frac{1}{2\sqrt{3}} \\ \frac{1}{2\sqrt{3}} \end{pmatrix} (1, 2, 3) = \begin{pmatrix} \frac{1}{2\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{\sqrt{3}}{2} \\ \frac{1}{2\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{\sqrt{3}}{2} \end{pmatrix}.$$

It is easily verified that $B_{\text{new}} K = BF_{\text{old}}$.

Assumption 4.2. *The sets* $\{\lambda_1, \ldots, \lambda_n\}$ *and* $\{\mu_1, \ldots, \mu_p\}$ *are closed under complex conjugation and disjoint.*

Justification: The closeness under complex conjugation of the above sets is necessary to guarantee that the closed-loop eigenvalues are self-conjugate, since any closed-loop system designed with the feedback that has physical sense (that is, real feedback) must have self-conjugate spectrum.

If $\{\mu_1, \ldots, \mu_p\} \cap \{\lambda_1, \ldots, \lambda_p\} \neq \emptyset$, it means that some open-loop eigenvalues that we have selected to reassign in fact would not move. In this case, we should renumber the open-loop eigenvalues in such a way that the eigenvalues that would remain unaltered would go last and the number p of the eigenvalues to be reassigned will be decreased. This way we obtain the partial eigenvalue assignment problems with $\{\mu_1, \ldots, \mu_p\} \cap \{\lambda_1, \ldots, \lambda_p\} = \emptyset$.

Designing a closed-loop system such that $\{\mu_1, \ldots, \mu_p\} \cap \{\lambda_{p+1}, \ldots, \lambda_n\} \neq \emptyset$ is generally considered a "bad practice" in engineering. Systems with such artificially created multiple eigenvalues are usually less robust compared to the systems designed with slightly perturbed $\mu_1, \ldots, \mu_p$ because multiple eigenvalues are usually very sensitive to perturbations.

The following theorem gives a parametric solution to the first-order partial eigenvalue assignment problem.

Theorem 4.1. (Parametric Solution to the Partial Eigenvalue Assignment Problem).
Let the Assumptions 4.1 *and* 4.2 *hold and let the pair* (A, B) *be partially controllable with respect to* $\{\lambda_1, \ldots, \lambda_p\}$*. Assume further that the closed-loop matrix has a complete set of eigenvectors. Let* $\Gamma = (\gamma_1, \ldots, \gamma_p)$ *be a matrix such that*

$$\gamma_j = \overline{\gamma_k} \text{ whenever } \mu_j = \overline{\mu_k}\,. \tag{4.4}$$

Set $\Lambda_1 = \mathrm{diag}(\lambda_1, \ldots, \lambda_p)$ *and* $\Lambda_{c1} = \mathrm{diag}(\mu_1, \ldots, \mu_p)$*. Let* Z_1 *be a unique nonsingular solution of the Sylvester equation*

$$\Lambda_1 Z_1 - Z_1 \Lambda_{c1} = Y_1^H B \Gamma\,. \tag{4.5}$$

Let Φ *satisfy the linear system*

$$\Phi Z_1 = \Gamma\,. \tag{4.6}$$

Then the real feedback matrix F *is given by*

$$F = \Phi Y_1^H \tag{4.7}$$

solves the partial eigenvalue assignment problem for the pair (A, B)*.*

Conversely, if there exists a real feedback matrix F *of the form* (4.7) *that solves the partial eigenvalue assignment problem for the pair* (A, B)*, then the matrix* Φ *can be constructed satisfying* (4.4) *through* (4.6)*.*

Proof. First, we prove the "converse part" of the Theorem. Let a matrix F of the form (4.7) solve the partial eigenvalue assignment problem. Denote by $X_{c1} = (x_{c1}, \ldots, x_{cp})$ the matrix of right eigenvectors of the closed-loop pencil corresponding to the eigenvalues $\mu_1, \ldots, \mu_p$. Define the matrix $\Gamma = FX_{c1}$. Then the following equation is obviously satisfied:

$$AX_{c1} - X_{c1}\Lambda_{c1} = B\Gamma\,. \tag{4.8}$$

Multiplying this equation to the left by the Y_1^H and defining $Z_1 = Y_1^H X_{c1}$, we obtain (4.5).

From (4.7) and (4.8), we have

$$0 = (A - B\Phi Y_1^H)X_{c1} - X_{c1}\Lambda_{c1} = B\Gamma - B\Phi Y_1^H X_{c1} = B(\Gamma - \Phi Z_1)\,. \tag{4.9}$$

Since B has linearly independent columns, (4.9) is equivalent to (4.6).

Finally, if $\mu_j = \overline{\mu_k}$, then $x_{cj} = \overline{x_{ck}}$, where $X_{c1} = (x_{c1}, \ldots, x_{cp})$. Since F is real and the j^{th} column of Γ is $\gamma_j = Fx_{cj}$, we get $\gamma_j = \overline{\gamma_k}$, proving (4.4).

Now we will prove the theorem in the other direction. Let Γ be chosen to satisfy (4.4) and (4.5). Since $\{\mu_1, \ldots, \mu_p\} \cap \{\lambda_1, \ldots, \lambda_p\} = \emptyset$, then Φ is also uniquely defined by (4.5) and (4.6).

Since Y_1 and X_2 are, respectively, the left and the right eigenvectors of A corresponding to disjoint sets of eigenvectors, we have $Y_1^H X_2 = 0$. Thus, for any Φ with $F = \Phi Y_1^H$ we have

$$(A - BF)X_2 = AX_2 - B\Phi(Y_1^H X_2) = X_2\Lambda_2\,, \tag{4.10}$$

where $\Lambda_2 = \operatorname{diag}(\lambda_{p+1}, \ldots, \lambda_n)$ and X_2 is the matrix of right eigenvectors of A corresponding to the eigenvalues $\lambda_{p+1}, \ldots, \lambda_n$. Thus, both the eigenvalues $\lambda_{p+1}, \ldots, \lambda_n$ and the associated right eigenvectors $x_{p+1}, \ldots, x_n$ of the closed-loop system are the same as those of the open-loop system.

It thus remains to be shown that with our above choice of Φ, the set $\{\mu_1, \ldots, \mu_p\}$ is also in the spectrum of $A - BF$ and the matrix F is real.

Since the set $\{\mu_1, \ldots, \mu_p\}$ and the spectrum of A are disjoint, the Sylvester equation (4.8) has a unique solution (see [4] for details), which we denote by X_{c1}. Multiplying the equation (4.8) by Y_1^H and noting that $Y_1^H A = \Lambda_1 Y_1^H$, we obtain

$$\Lambda_1(Y_1^H X_{c1}) - (Y_1^H X_{c1})\Lambda_{c1} = Y_1^H B\Gamma\,. \tag{4.11}$$

Thus, $Y_1^H X_{c1}$ and Z_1 satisfy the same Sylvester equation. Since this Sylvester equation has a unique solution (because spectra of Λ_1 and Λ_{c1} are disjoint), we have

$$Z_1 \;=\; Y_1^H X_{c1}\,. \tag{4.12}$$

Using (4.6) and (4.12), we obtain

$$(A - BF)X_{c1} - X_{c1}\Lambda_{c1} = AX_{c1} - X_{c1}\Lambda_{c1} - B\Phi Y_1^H X_{c1} =$$
$$B(\Gamma - \Phi(Y_1^H X_{c1})) = 0,$$

which shows that the set $\{\mu_1, \dots, \mu_p\}$ is in the spectrum of $A - BF$.

To complete the proof of the theorem, we must show that F is real.

Since the set $\{\mu_1, \dots, \mu_p\}$ is closed under complex conjugation, there exists a permutation matrix T_c such that $\overline{\Lambda_{c1}} = T_c^T \Lambda_{c1} T_c$. Then (4.4) implies that $\overline{\Gamma} = \Gamma T_c$. Similarly, there exists a permutation matrix T such that $\overline{\Lambda_1} = T^T \Lambda_1 T$, $\overline{X_1} = X_1 T$ and $\overline{Y_1} = Y_1 T$. Conjugating the equation (4.5), we get

$$(T^T \Lambda_1 T)\overline{Z_1} - \overline{Z_1}(T_c^T \Lambda_{c1} T_c) = (T^T Y_1^H) B (\Gamma T_c)\,. \tag{4.13}$$

Clearly $\overline{Z_1} = T^T Z_1 T_c$, since such $\overline{Z_1}$ satisfies the equation (4.13), because $T_c^T = T_c^{-1}$. Again, conjugating (4.6), we get

$$\overline{\Phi}(T^T Z_1 T_c) = \Gamma T_c\,,$$

which implies that $\overline{\Phi} = \Phi T$.

Therefore,

$$\overline{F} = (\Phi T)(T^T Y_1^H) = F\,,$$

showing that the obtained feedback matrix F is real. □

Remark 4.1. *Substituting the expression of Γ from (4.6) into (4.5), we obtain*

$$(\Lambda_1 - Y_1^H B \Phi) Z_1 = Z_1 \Lambda_{c1}\,, \tag{4.14}$$

which shows that Z_1 is the eigenvector matrix for $\Lambda_1 - Y_1^H B \Phi$. From (4.14) it then follows that the nonsingularity of Z_1 is equivalent to the linear independence of the eigenvectors of the closed-loop matrix $A - BF$.

This observation is important because it is well known that the sensitivity of the eigenvalues of the closed-loop matrix is related to the conditioning of the eigenvector matrix.

Based on the Theorem 4.1, we now state the following algorithm:

Algorithm 4.2. (Parametric Algorithm for Partial Eigenvalue Assignment Problem).

Inputs:

(a) *The $n \times n$ matrix A.*

(b) *The $n \times m$ control matrix B.*

(c) *The set $\{\mu_1, \dots, \mu_p\}$, closed under complex conjugation.*

(d) *The self-conjugate subset $\{\lambda_1, \dots, \lambda_p\}$ of the spectrum $\{\lambda_1, \dots, \lambda_n\}$ of the matrix A and the associated right eigenvector set $\{y_1, \dots, y_p\}$.*

Outputs:

The real feedback matrix F such that the spectrum of the closed-loop matrix $A - BF$ is $\{\mu_1, \ldots, \mu_p; \lambda_{p+1}, \ldots, \lambda_n\}$.

Assumptions:

(a) The matrix pair (A, B) is partially controllable with respect to the eigenvalues $\lambda_1, \ldots, \lambda_p$.

(b) The sets $\{\lambda_1, \ldots, \lambda_p\}$, $\{\lambda_{p+1}, \ldots, \lambda_n\}$, and $\{\mu_1, \ldots, \mu_p\}$ are disjoint.

Step 1. *Form*

$$\Lambda_1 = \mathrm{diag}(\lambda_1, \ldots, \lambda_p),\ Y_1 = (y_1, \ldots, y_p),\ \textit{and}\ \Lambda_{c1} = \mathrm{diag}(\mu_1, \ldots, \mu_p).$$

Step 2. *Choose arbitrary $m \times 1$ vectors $\gamma_1, \ldots, \gamma_p$ in such a way that $\overline{\mu_j} = \mu_k$ implies $\overline{\gamma_j} = \gamma_k$ and form $\Gamma = (\gamma_1, \ldots, \gamma_p)$.*

Step 3. *Find the unique solution Z_1 of the Sylvester equation*

$$\Lambda_1 Z_1 - Z_1 \Lambda_{c1} = Y_1^H B\Gamma\,.$$

If Z_1 is ill-conditioned, then return to Step 2 and select different $\gamma_1, \ldots, \gamma_p$.

Step 4. *Solve $\Phi Z_1 = \Gamma$ for Φ.*

Step 5. *Form $F = \Phi Y_1^H$.*

5 Numerical Examples

In this section, we report results of our numerical experiments with Algorithm 4.2 on a 400×400 matrix obtained by discretization of the partial differential equation

$$\frac{\partial u}{\partial t} \quad = \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + 20\frac{\partial u}{\partial x} + 180u(x, y, t) + \sum_{i=1}^{2} F_i(x, y) g_i(t) \tag{5.15}$$

on the unit square $\Omega = (0, 1) \times (0, 1)$ with the Dirichlet boundary conditions

$$u(x, y, t) = 0 \text{ for } (x, y) \in \partial\Omega \text{ and } t \geq 0$$

and some initial condition which is of no importance for the PEVA problem. This problem was earlier considered by [6]. Using finite difference scheme of order $O(||\Delta x||^2, ||\Delta y||^2)$ we discretize (5.15) in the region Ω with 20 interior points in both the x and y directions, thus obtaining 400×400 matrix problem of the form (1.1). The 400×2 matrix B, whose i-th column discretizes the function $F_i(x, y)$ is filled with random numbers between -1 and 1.

Using sparse MATLAB command `eigs`, the following ten eigenvalues with the largest real parts are computed

$$\lambda_1 = 55.0660, \lambda_2 = 29.2717, \lambda_3 = 25.7324, \lambda_4 = -0.0618, \lambda_5 = -13.0780,$$
$$\lambda_6 = -22.4283, \lambda_7 = -42.4115, \lambda_8 = -48.2225, \lambda_9 = -71.0371, \lambda_{10} = -88.3402.$$

The residual of each eigenpair $||y^H(A - \lambda I)|| < 4 \cdot 10^{-12}$ and each left eigenvector is normalized.

Algorithm 4.2 is used to reassign $\lambda_1, \lambda_2, \lambda_3$ and λ_4 to $-7, -8, -9$ and -10, respectively, obtaining the 2×400 feedback matrix F with $||F||_2 < 127$. Note that the $||A||_2 = 3.3 \cdot 10^3$. The ten eigenvalues of the closed-loop matrix $A - BF$ with the largest real parts obtained by the algorithm are the following:

$$\mu_1 = -7.0000, \mu_2 = -8.0000, \mu_3 = -9.0000, \mu_4 = -10.0000, \lambda_5 = -13.0780,$$
$$\lambda_6 = -22.4283, \lambda_7 = -42.4115, \lambda_8 = -48.2225, \lambda_9 = -71.0371, \lambda_{10} = -88.3402.$$

6 Conclusion

The design of large space structures, power plants, computer networks, etc., give rise to large-scale control problems. For most of these problems, in practice, only a small number of system eigenvalues are "bad" in the sense that they are not in the stability region or other desirable region of the complex plane. Thus it make sense to reassign only those small number of bad eigenvalues, keeping the remaining large number of good eigenvalues invariant. The partial eigenvalue assignment problem is thus a practically significant problem.

New necessary and sufficient conditions for existence and uniqueness of solution for this problem has been derived and a parametric algorithm for numerical solution has been presented in this paper. The algorithm requires knowledge of only those small number of eigenvalues (and the associated eigenvectors) that are required to be reassigned, and the major computational requirements are solutions of a small Sylvester equation and a linear algebraic system, for which there exist excellent numerical methods. The algorithm is thus numerically viable and computationally feasible for large and sparse problems. Furthermore, the parametric nature of the algorithm offers an opportunity to design a numerically robust feedback controller.

A similar parametric approach has been also developed for the partial eigenstructure assignment problem in the recent Ph.D. Thesis [7].

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