PARAMETER TUNING OF A NON INTEGER ORDER PID CONTROLLER

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Abstract

In this paper a new type of PID controllers is introduced and some properties are given. The novelty of the proposed controllers consists in the extension of derivation and integration order from integer to non integer numbers. This approach provides a more flexible tuning strategy and therefore an easier achieving of control requirements with respects to classical controllers.

1 Introduction

The idea of fractional derivatives and integrals seems to be quite a strange topic, very hard to explain, due to the fact that, unlike commonly used differential operators, it is not related to some important geometrical meaning, such as the trend of functions or their convexity.

A number N of first order differential equations usually model a complex dynamic. In systems theory, we call N the degree of the system. Moreover, the theory of Laplace transformation in the case of linear systems has given us the possibility of studying an input–output relation via the ratio of its Laplace transform, which is called the "Transfer Function" of the system and whose denominator is a polynomial of degree N. In fact there are physical phenomena whose study involves transfer functions of degree m, m being a *non integer* number.

For this reason, this mathematical tool could be judged "far from reality". But many physical phenomena have "intrinsic" fractional order description and so fractional order calculus is necessary in order to explain them.

Transmission lines [1], electrical noises [2-3], dielectric polarization [4] and heat transfer phenomena [5] are some of the fields having "Non Integer Order" physical laws.

2 An overview on non integer order systems

The most common definition of non integer order integral is the following [6-7]:

$$\frac{d^{-q}f(t)}{dt^{-q}} = \frac{1}{\Gamma(q)}\int_0^t (t-\tau)^{q-1}f(\tau)d\tau \tag{2.1}$$

The lower limit is chosen to be zero (it could be any real number) because in the following time series with $t > 0$ will be considered. In the above definition, Γ(q) is the factorial function, defined for positive real q, by the following expression:

$$\Gamma(q) = \int_0^\infty x^{q-1}e^{-x}dx \tag{2.2}$$

for which, when q is an integer, it holds that:

$$\Gamma(q+1) = q! \tag{2.3}$$

The definition of fractional derivative easily derives from (2.1) by taking an n order derivative (n suitable integer) of an m order integral (m suitable *non* integer) to obtain an $n - m = q$-order one:

$$\frac{d^q f(t)}{dt^q} = \frac{d^{n-m} f(t)}{dt^{n-m}} = \frac{d^n}{dt^n}\left[\frac{d^{-m} f(t)}{dt^{-m}}\right] == \frac{1}{\Gamma(m)}\frac{d^n}{dt^n}\int_0^t (t-\tau)^{m-1} f(\tau)d(\tau) \tag{2.4}$$

It is now useful to introduce the most important features of fractional systems. They will be discussed using the same tools usually adopted for integer order systems which allow an easy comparison among the two different behaviors. For example, let us focus on Bode Diagrams, that is the principal tool in systems and control theory: considering the following equation:

$$F(s) = \frac{k}{\left(\frac{s}{p}+1\right)^m} \tag{2.5}$$

and assuming s=$j\omega$ we obtain:

$$\begin{aligned} F(j\omega) = \left[\tfrac{k^{1/m}}{(j\omega/p+1)}\right]^m = \left[\left|\tfrac{k^{1/m}}{(j\omega/p+1)}\right| e^{j\varphi\left[\frac{k^{1/m}}{(j\omega/p+1)}\right]}\right]^m = \\ = \left|\tfrac{k^{1/m}}{(j\omega/p+1)}\right|^m e^{jm\varphi\left[\frac{k^{1/m}}{(j\omega/p+1)}\right]} \end{aligned} \tag{2.6}$$

and, therefore the magnitude expressed in decibels is

$$\begin{aligned} |F(j\omega)|_{dB} = 20\log_{10}\left[\tfrac{k^{1/m}}{\sqrt{\omega^2/p^2+1}}\right]^m = \\ 20\log_{10}k - 20m\log_{10}\sqrt{\omega^2/p^2+1} \end{aligned} \tag{2.7}$$

It must be noted that, if $\omega \to \infty$, (2.7) becomes $-20m\log_{10}(\omega/p)$ resulting, on a semi-logarithmic plane, in a line having slope $-20m$ dB/dec (instead of 20 dB/dec for first order systems). This fact is useful to plot an asymptotic diagram whose maximum error e_{max} can be found close to the pole ω=p. This error can be calculated as follows:

$$\begin{aligned} e_{\max} = \left||F(jp)|_{dB,app} - |F(jp)|_{dB}\right| = \\ \left|-20m\log_{10}\tfrac{\omega}{p}\Big|_{\omega=p} - \left(-20m\log_{10}\sqrt{\tfrac{\omega^2}{p^2}+1}\right)_{\omega=p}\right| \cong 3m\,dB \end{aligned} \tag{2.8}$$

while for first order systems this value is 3 dB. Examples of magnitude Bode's diagrams are reported in Fig. 2.1.

Remark: It is quite evident that the fractional order m modulates the slope of the magnitude diagram, providing a parameter useful for the open loop synthesis of the controller.

Regarding the phase displacement, it must be noted that, considering the exponent of expression 2.6, it holds that:

$$\varphi\left[F(j\omega)\right] = m\varphi\left[\frac{k^{1/m}}{j\omega/p+1}\right] = m\varphi\left[\frac{1}{j\omega/p+1}\right] = -marctg\frac{\omega}{p} \tag{2.9}$$

Expression (2.9) shows that m modulates the scale of the phase law. In fact, it can be easily seen that for $\omega \to \infty$the phase angle approaches $-m\pi/2$ instead of $-\psi/2$ typical of first order systems. Examples of phase diagrams are depicted in Fig. 2.2.

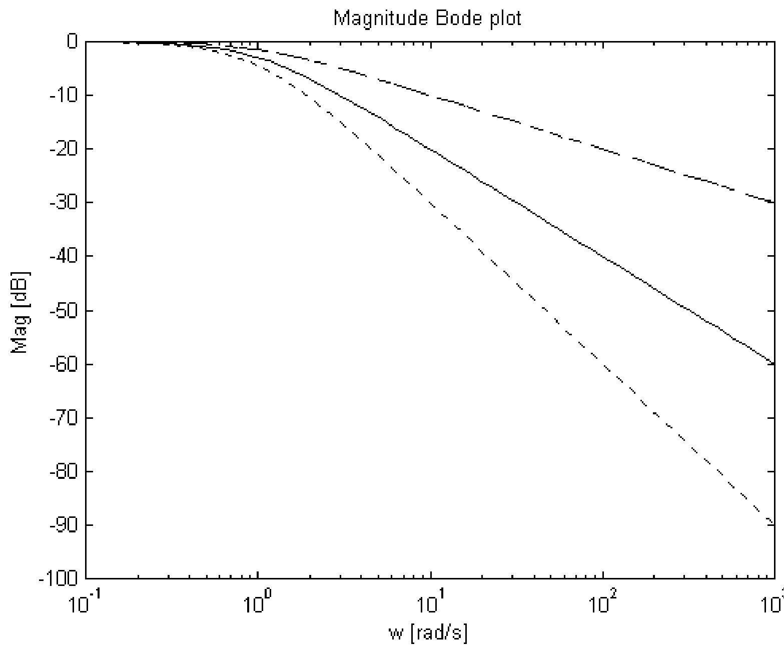

Fig. 1. Magnitude Bode Plot of fractional systems$F(s)$ =$1/(s+1)^m$ with m =1 (solid), m =0.5 (dashed), m =1.5 (dotted).

Similar results can be obtained considering the transfer function

$$F(s) = \frac{k}{\left(\frac{s}{p}\right)^m + 1} \tag{2.10}$$

For large ω, the magnitude in dB is

$$
\begin{aligned}
|F(j\omega)|_{dB} &= 20\log_{10}\frac{k}{\sqrt{\omega^{2m}/p^{2m}+2\omega^m/p^m\cos(m\pi/2)+1}} \approx \\
&\approx 20\log_{10}k - 20m\log_{10}(\omega/p)
\end{aligned}
\tag{2.11}
$$

It must be noted that in this case that the maximum error e_{max} at $\omega{=}p$ could be zero for a certain value of m. This fact happens when

$$
\begin{aligned}
&2\cos(m\pi/2)+1=0 \\
&(m=4/3, \quad m=8/3)
\end{aligned}
\tag{2.12}
$$

The general formula is the following:

$$
\begin{aligned}
e_{\max} &= \Big| |F(jp)|_{dB,app} - |F(jp)|_{dB} \Big| = \\
&= 10\log_{10}\left(2+2\cos\frac{m\pi}{2}\right) = 10\log_{10}4\left(\frac{1+\cos\frac{m\pi}{2}}{2}\right) \cong 6+20\log_{10}\left|\cos\frac{m\pi}{4}\right|
\end{aligned}
\tag{2.13}
$$

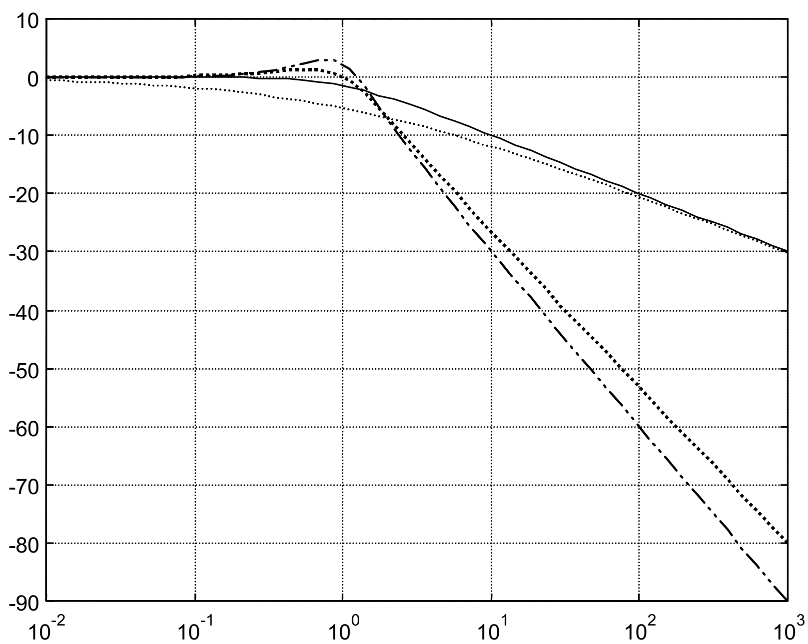

Fig. 2. For case 1 the formula to evaluate the correction error is $e = 3m$db, giving $e = 1.5$db. For all other cases the formula is $e = 6 + 20 * log_{10}(|cos(m\pi/4)|$db, resulting $e = 5.3$db, $e = -2.34$db and $e = 0$db respectively. $1/(s+1)^{0.5}$ continuous, $1/(s^{0.5}+1)$ small dot, $1/(s^{1.5}+1)$ dashed and $1/(s^{4/3}+1)$ big dot.

3 The non integer order PID controller

Suppose to have a non integer order PID of the form

$$C(s) = K_P + \frac{K_I}{s^\lambda} + K_D s^\lambda \tag{3.14}$$

While a traditional PID provides a phase contribution $-\pi/2 < \theta < \pi/2$, the PI$^\lambda$D$^\lambda$ provides $-\lambda\pi/2 < \theta < \lambda\pi/2$, with a significant improvement when $\lambda > 1$

Let us now consider the frequency domain design. According to project requirements, C(s) must be so that

$$\begin{array}{l} C(j\omega_1)G(j\omega_1) = e^{j(\varphi_m - \pi)} \quad (\varphi_m = \text{phase margin},\ \omega_1 = 0\text{db frequency}) \\ C(j\omega_1) = \frac{e^{j(\varphi_m-\pi)}}{|G(j\omega_1)|e^{j\arg(G(j\omega_1))}} \\ K_P + \frac{K_I e^{-j\frac{\lambda\pi}{2}}}{\omega_1^\lambda} + K_D\omega_1^\lambda e^{j\frac{\lambda\pi}{2}} = \frac{e^{j\theta}}{|G(j\omega_1)|} \quad (\theta = \varphi_m - \pi - \arg(G(j\omega_1))) \end{array} \tag{3.15}$$

The following two equations in the four variables K$_P$,K$_I$,K$_D$ and λgan be written:

$$\begin{cases} K_P + \left(\frac{K_I}{\omega_1^\lambda} + K_D\omega_1^\lambda\right)\cos\frac{\lambda\pi}{2} = \frac{\cos\theta}{|G(j\omega_1)|} \\ \left(-\frac{K_I}{\omega_1^\lambda} + K_D\omega_1^\lambda\right)\sin\frac{\lambda\pi}{2} = \frac{\sin\theta}{|G(j\omega_1)|} \end{cases} \tag{3.16}$$

The use of non integer order PID provide to the controller a further tuning parameter, which improves the set of obtainable configurations

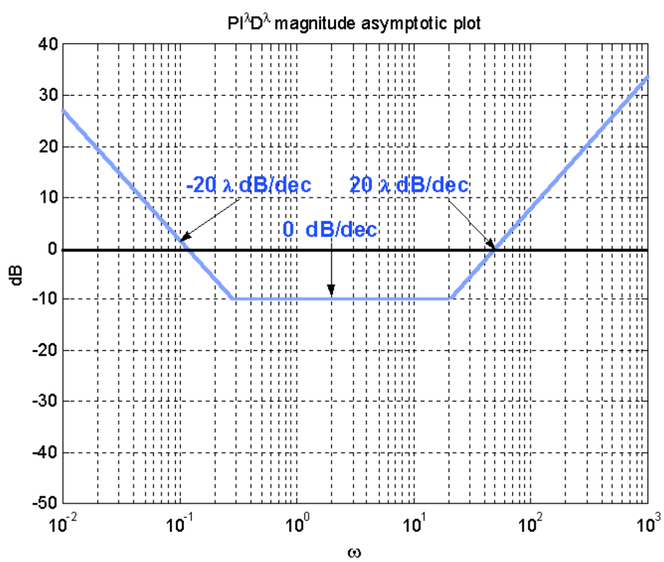

Fig. 3. An example of PI$^\lambda$D$^\lambda$ magnitude asymptotic plot

The transfer function of a PI$^\lambda$D$^\lambda$ controller is the following:

$$\begin{array}{l} C(s) = K_P + \frac{K_I}{s^\lambda} + K_D s^\lambda = \frac{K_D s^{2\lambda} + K_P s^\lambda + K_I}{s^\lambda} = \\ = \frac{K_D(s^\lambda + a_1)(s^\lambda + a_2)}{s^\lambda} \qquad where \quad a_{1,2} = \frac{K_P \pm \sqrt{K_P^2 - K_I K_D}}{2K_D} \end{array} \tag{3.17}$$

4 Conclusion

In this paper the non integer order $PI^{\lambda}D^{\lambda}$ controller have been introduced and a new tuning strategy has been presented. The approach is validate step by step by an extension of the classical PID control theory. Further research activities are running in order to define new effective tuning techniques for non integer order controller of the more general for $PI^{\lambda}D^{\mu}$.

5 Acknowledgement

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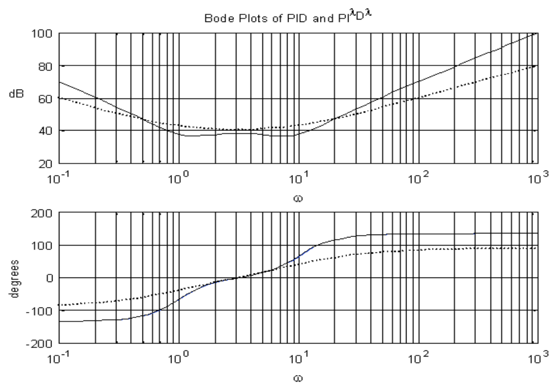

Fig. 4. An example of PID (dotted) and PI$^{\lambda}$D$^{\lambda}$, magnitude and phase plot.

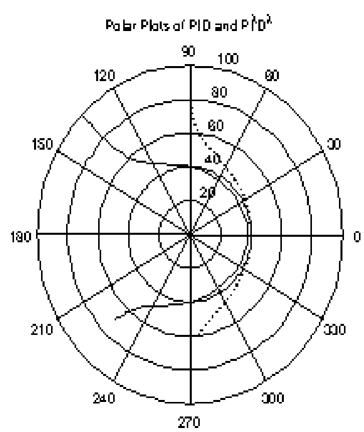

Fig. 5. An example of PID (dotted) and PI$^{\lambda}$D$^{\lambda}$, magnitude and polar plot.

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