On Rosenbrock models over a commutative ring

Vakhtang Lomadze
Institute of Mathematics
Tbilisi 380093, Georgia

Abstract

Rosenbrock's notion of system equivalence is general in nature; it is a kind of equivalence which in algebra is often termed as stable. We have shown recently that Fuhrmann's notion of system equivalence can be viewed as a homotopy equivalence, and as such is also general in nature. This note deals with a generalization of the theory of system equivalences from the field case to the commutative ring case.

In this note we generalize the classical theory of Rosenbrock models and their equivalences, given in [2,6,7], to the commutative ring case. We shall follow very closely our recent paper [3]. For other generalization the reader is refered to [5].

Throughout, A is an arbitrary commutative ring, s an indeterminate, m an input number and p an output number.

We define $A(s)$ to be the localization of $A[s]$ with respect to polynomials with invertible leading coefficient. Elements of this ring will be called rational functions. A rational function f/g will be said to be proper if $deg(f) \leq deg(g)$. We let O denote the ring of proper rational functions. One has the notion of a finite $A[s]$-module, the notions of left and right coprimeness. (For details, see [4].)

A Rosenbrock model is a quintuple $(Z; T, U, V, W)$, where Z is a finitely generated A-module, $T : Z[s] \to Z[s]$ is a "generical" isomorphism and $U : A[s]^m \to Z[s]$, $V : Z[s] \to A[s]^p$, $W : A[s]^m \to A[s]^p$ are arbitrary homomorphisms. (The condition on T means that it induces an isomorphism $Z(s) \simeq Z(s)$.) The transfer function is defined as the rational matrix $VT^{-1}U + W$. A model is called regular if its transfer function is proper.

A transformation of a model $(Z; T, U, V, W)$ into a model $(Z'; T', U', V', W')$ is a quadruple (K, L, M, N) consisting of homomorphisms $K : Z[s] \to Z'[s]$, $L : A[s]^m \to Z'[s]$, $M : Z[s] \to Z'[s]$, and $N : Z[s] \to A[s]^p$ such that

$$\begin{bmatrix} M & 0 \\ N & I \end{bmatrix} \begin{bmatrix} T & U \\ -V & W \end{bmatrix} = \begin{bmatrix} T' & U' \\ -V' & W' \end{bmatrix} \begin{bmatrix} K & -L \\ 0 & I \end{bmatrix};$$

that is,

$$MT = T'K, \ MU = -T'L + U', \ NT - V = -V'K, \ NU + W = V'L + W'.$$

If $\Phi_1 = (K_1, L_1, M_1, N_1)$ and $\Phi_2 = (K_2, L_2, M_2, N_2)$ are two transformations such that the range of the first one is equal to the domain of the second, then their composition is defined to be

$$\Phi_2 \circ \Phi_1 = (K_2K_1, K_2L_1 + L_2, M_2M_1, N_2M_1 + N_1).$$

The identity transformation of a Rosenbrock model $\Sigma = (Z; T, U, V, W)$ is defined as $I_\Sigma = (I, 0, I, 0)$.

One can easily check that Rosenbrock models together with transformations form a category. Isomorphisms in this category are called strict equivalences. It is easy to see that a transformation (K, L, M, N) is a strict equivalence if and only if K and M are unimodular.

Proposition 1. *Two Rosenbrock models have the same transfer function if there is a transformation of one into the other.*

Proof. Left to the reader. (See also [3].) □

A Kalman model is a quintuple $(X; F, G, H, J)$, where X is a finitely generated A-module, $F : X \to X$, $G : A^m \to X$ and $H : X \to A^p$ are homomorphisms, and J is a polynomial $p \times m$ matrix. Such a quintuple can be rewritten as $(X; sI - F, G, H, J)$, and so, a Kalman model can be viewed as a special form of a Rosenbrock model. One introduces in an evident way the notion of transformations for Kalman models. Obviously Kalman models and their transformations form a category. Isomorphic Kalman models are said to be similar. As in the classical case, there is an equivalent definition of a Kalman model (see [4]), which will be used in the sequel. This can be defined as a quadruple $(Q; \varphi, \psi, J)$, where Q is a finite $A[s]$-module, $\varphi : A[s]^m \to Q$ and $\psi : Q \to A(s)^p/A[s]^p$ are homomorphisms, and J is as above.

The problem is to define a procedure that could allow one to bring an arbitrary Rosenbrock model to the Kalman form (see [2,6,7]).

Given a Rosenbrock model $\Sigma = (Z; T, U, V, W)$, we define the Kalman representation $KR(\Sigma)$ as the quadruple consisting of the module $Z[s]/TZ[s]$, the homomorphisms

$$u \mapsto (Uu) \bmod TZ[s] \ (u \in A[s]^m),$$

$$z \bmod TZ[s] \mapsto (VT^{-1}z) \bmod A[s]^p \ (z \in Z[s]),$$

and the polynomial part of the transfer function of Σ. Notice that $KR(\Sigma)$ has the same transfer function as Σ.

One can see that KR is a (covariant) functor from the category of Rosenbrock models to that of Kalman models, and one can check without difficulty that KR preseves the properties of regularity, controllability and observability.

Let $\Sigma = (Z; T, U, V, W)$ and $\Sigma' = (Z'; T', U', V', W')$ be Rosenbrock models. If $\Phi_1 = (K_1, L_1, M_1, N_1)$ and $\Phi_2 = (K_2, L_2, M_2, N_2)$ are transformations of the first one into the second, then we say that Φ_1 is homotopic to Φ_2 (and write $\Phi_1 \approx \Phi_2$) if there exists a homomorphism $H : Z[s] \to Z'[s]$ satisfying the following two equivalent conditions

$$T'H = M_1 - M_2 \quad \text{and} \quad HT = K_1 - K_2.$$

Lemma 1. *Let $\Sigma = (Z; T, U, V, W)$ be a Rosenbrock model. Then, all transformations that are homotopic to I_Σ have the form*

$$(HT + I, -HU, TH + I, -VH),$$

where $H \in Hom(Z[s], Z[s])$.

Proof. Left to the reader. □

Lemma 2. *Homotopy is an equivalence relation on the set of all transformations of one Rosenbrock model into another.*

Proof. Left to the reader. (See also [3].) □

A transformation $\Phi : \Sigma \to \Sigma'$ is a homotopy equivalence if there exists a transformation $\Phi' : \Sigma' \to \Sigma$ such that

$$\Phi' \circ \Phi \approx I_\Sigma \quad \text{and} \quad \Phi \circ \Phi' \approx I_{\Sigma'}.$$

(Compare with the definition of strict equivalence: A transformation Φ is a strict equivalence if there exists a transformation Φ' such that $\Phi' \circ \Phi = I_\Sigma$ and $\Phi \circ \Phi' = I_{\Sigma'}$.) Intuitively, the notion of homotopy equivalence is a notion that is somewhat coarser than the notion of strict equivalence.

Two Rosenbrock models are homotopy equivalent (or equivalent in the sense of Fuhrmann) if there exists a homotopy equivalence between them. We shall write $\Sigma \approx \Sigma'$ to denote that Σ and Σ' are homotopy equivalent.

Lemma 3. *The relation between Rosenbrock models of being homotopy equivalent is an equivalence relation.*

Proof. Follows from the previous lemma. □

Proposition 2. *Two transformations are homotopy equivalent if and only if they have equal Kalman representations.*

Proof. Left to the reader. (See also [3].) □

From this proposition we immediately obtain the following

Theorem 1. *Every Rosenbrock model is homotopy equivalent to a Kalman model; two Kalman models are homotopy equivalent if and only if they are similar.*

For each $r \geq 1$, let Ω^r denote the Rosenbrock model $(A^r; I_r, 0, 0, 0)$. (We put $\Omega^0 = 0$.) If $(Z_1; T_1, U_1, V_1, W_1)$ and $(Z_2; T_2, U_2, V_2, W_2)$ are Rosenbrock models, their parallel connection is defined to be

$$(Z_1 \oplus Z_2; \begin{bmatrix} T_1 & 0 \\ 0 & T_2 \end{bmatrix}, \begin{bmatrix} U_1 \\ U_2 \end{bmatrix}, [V_1 \ V_2], W_1 + W_2).$$

We say that two Rosenbrock models Σ and Σ' are stably equivalent (or equivalent in the sense of Rosenbrock) if

$$\Omega^l \oplus \Sigma \simeq \Omega^{l'} \oplus \Sigma'$$

for some nonnegative integers l and l'.

We need the following

Lemma 4. *A Rosenbrock model Σ with "latent" rank $r \geq 1$ is homotopy equivalent to 0 if and only if it is strictly equivalent to Ω^r.*

Proof. Left to the reader. (See also [3].) $\square$

Theorem 2. *Two Rosenbrock models are stably equivalent if and only if they are homotopy equivalent.*

Proof. Let $\Sigma = (Z; T, U, V, W)$ and $\Sigma' = (Z'; T', U', V', W')$ be Rosenbrock models.

"If" Let (K, L, M, N) be a homotopy equivalence. Then, there exist a transformation (K', L', M', N') and homomorphisms H, H' such that

$$(K'K, K'L + L', M'M, N'M + N) = (HT + I, -HU, TH + I, -VH)$$

and

$$(KK', KL' + L, MM', NM' + N') = (H'T' + I, -H'U', T'H' + I, -V'H').$$

Now, from the fact that (K, L, M, N) is a transformation we have

$$\begin{bmatrix} M' & I & 0 \\ -T'H' & -M & 0 \\ V'H' & -N & -I \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ 0 & T & U \\ 0 & -V & W \end{bmatrix} =$$

$$\begin{bmatrix} I & 0 & 0 \\ 0 & T' & U' \\ 0 & -V' & W' \end{bmatrix} \begin{bmatrix} M' & T & U \\ -H' & -K & L \\ 0 & 0 & -I \end{bmatrix}.$$

The extreme matrices here are unimodular. Indeed, by the equalities above, we have

$$\begin{bmatrix} M' & I \\ -T'H' & -M \end{bmatrix} \begin{bmatrix} M & I \\ -TH & -M' \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}.$$

and

$$\begin{bmatrix} M' & T \\ -H' & -K \end{bmatrix} \begin{bmatrix} M & T' \\ -H & -K' \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}.$$

"Only if" Using the previous lemma, we have

$$\Sigma \approx \Omega^l \oplus \Sigma \simeq \Omega^{l'} \oplus \Sigma' \approx \Sigma'.$$

The theorem is proved. $\square$

The definitions of poles and zeros as given in [1,8] (and reproduced in [3]) can be obviously generalized. Indeed, let $\Sigma = (Z; T, U, V, W)$ be a Rosenbrock model, and let G be its transfer function. Let "T" denote the functor taking the torsion part. We define the finite and infinite pole modules to be

$$\mathcal{P}_f = \frac{Z[s]}{TZ[s]} \quad \text{and} \quad \mathcal{P}_\infty = \frac{O^p + GO^m}{O^p}.$$

We define the finite (invariant) and infinite (invariant) zero modules to be

$$\mathcal{Z}_f = \mathrm{T}\left(\frac{Z[s] \oplus A[s]^p}{\begin{bmatrix} T & U \\ -V & W \end{bmatrix} (Z[s] \oplus A[s]^m)}\right) \quad \text{and} \quad \mathcal{Z}_\infty = \mathrm{T}\left(\frac{O^p + GO^m}{GO^m}\right).$$

We define the input-decoupling and output-decoupling zero modules to be

$$\mathcal{Z}^{i.d.} = \frac{Z[s]}{TZ[s] + UA[s]^m} \quad \text{and} \quad \mathcal{Z}^{o.d.} = \mathrm{T}\left(\frac{Z[s] \oplus A[s]^p}{\begin{bmatrix} T \\ -V \end{bmatrix} Z[s]}\right).$$

(Decoupling zero modules at infinity are defined to be zero.) All these modules surely are finite. Remark that the model has no i.d. zeros if and only if T and U are left coprime, and has no o.d. zeros if and only if T and V are right coprime.

It is easily seen that the constructions of poles and zeros are functorial; that is, a transformation gives rise canonically to homomorphisms of pole modules and zero modules.

One can check without difficulty that two homotopy equivalent transformations induce the same homomorphisms of pole and zero modules. (The statement is trivial for the infinite poles and zeros, in view of Proposition 1.) Hence, we have the following

Theorem 3. *Pole and zero modules are invariant under homotopy equivalence.*

References

[1] H. Bourlès and M. Fliess, Finite poles and zeros of linear systems: an intrinsic approach, Int. J. Control, 68 (1997) 897-922.

[2] P. Fuhrmann, On strict system equivalence and similarity, Int. J. Control, 25 (1977) 5-10.

[3] V. Lomadze, Rosenbrock models and their homotopy equivalence, Linear Algebra Appl. (to appear).

[4] V. Lomadze, On Kalman models over a commutative ring, (these Proceedings).

[5] J.F. Pommaret and A. Quadrat, Equivalences of linear control systems, Proceedings of MTNS Symposium (Perpignan) 2000.

[6] H. H. Rosenbrock, *State Space and Multivariable Theory*, Nelson, London, 1970.

[7] H. H. Rosenbrock, The transformation of strict system equivalence, Int. J. Control, 25 (1977) 11-19.

[8] B.F. Wyman, M.K. Sain, G. Conte, and A.M. Perdon, On the poles and zeros of a transfer function, Linear Algebra Appl., 122/123/124 (1989), 123-144.