On Fliess models over a commutative ring

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Abstract

Fliess models come as natural generalizations of classical polynomial models (D, N). As known, they are defined in terms of finitely generated modules and provide an adequate description of arbitrary linear systems (not necessarily observable or controllable). In this note we extend Fliess' approach to linear systems defined over an arbitrary commutative ring. We believe that the exposition will be of interest even for the field case.

In this note we present a generalization of Fliess' module-theoretic approach, as developed in [1], to the commutative ring case.

Throughout, A is an arbitrary commutative ring, s an indeterminate, m an input number and p an output number. We shall denote by $A(s)$ the ring of rational functions and by O the ring of proper rational functions (see [4]).

A linear system is a quintuple $(X; F, G, H, J)$, where X is a finitely generated A-module $F : X \to X$, $G : A^m \to X$, $H : X \to A^p$ are A-linear maps, and J is a polynomial $p \times m$ matrix. The rational matrix $H(sI - F)^{-1} + J$ is called the transfer function. The reader is refered to [4] for the definitions of regularity, controllability and observability, and minimality.

We define a Fliess model as a triple $(M; \delta, \nu)$, where M is a finitely generated $A[s]$-module, $\delta : A[s]^p \to M$ a "generical" isomorphism and $\nu : A[s]^m \to M$ an arbitrary homomorphism. The condition on δ means that it induces an isomorphism $A(s)^p \simeq M \otimes A(s)$. This implies that the module M is quite specific; in particular, all its torsion elements are anihilated by monic polynomials.

A transformation of a Fliess model $(M; \delta, \nu)$ into a Fliess model $(M_1; \delta_1, \nu_1)$ is a homomorphism $\phi : M \to M_1$ such that $\delta_1 = \phi \circ \delta$ and $\nu_1 = \phi \circ \nu$. Obviously Fliess models together with transformations form a category.

Let $(M; \delta, \nu)$ be a Fliess model. The rational matrix $\nu\delta^{-1}$ is called the transfer function. The model is called controllable if $M = \delta A[s]^p + \nu A[s]^m$ and observable if M has no torsion. If the model is both controllable and observable it is said to be minimal.

Example 1. Let T be a transfer function. Then $(A[s]^p + TA[s]^m; id, T)$ is a minimal Fliess model. (Notice that every minimal Fliess model is obtained this way. Indeed, if $(M; \delta, \nu)$ is such a model and T is its transfer function, then $\delta^{-1}|_M$ is an isomorphism of M onto $A[s]^p + TA[s]^m$.)

Example 2. Let D be a polynomial $p \times p$ matrix such that $det(D)$ is monic, and let N be an arbitrary $p \times m$ polynomial matrix. Then $(A[s]^p, D, N)$ is an observable Fliess model. (But not every observable Fliess model is obtained this way.)

Lemma 1. *Let $(X; F, G, H, J)$ be a linear system. Set*

$$M = (X[s] \oplus A[s]^p) / \begin{bmatrix} sI - F \\ -H \end{bmatrix} X[s],$$

and define δ and ν respectively as the compositions

$$A[s]^p \to X[s] \oplus A[s]^p \to M \quad \text{and} \quad A[s]^m \to X[s] \oplus A[s]^p \to M.$$

Then $(M; \delta, \nu)$ is a Fliess model, and its transfer function coincides with that of $(X; F, G, H, J)$.

Proof. Consider the commutative diagram

$$\begin{array}{ccccccccc} 0 \to & X[s] & \to & X[s] \oplus A[s]^p & \to & M & \to 0 \\ & || & & \downarrow & & \downarrow & \\ 0 \to & X[s] & \to & X[s] & \to & X_F & \to 0 \end{array},$$

which has exact rows. Tensoring this by $A(s)$, we obtain a commutative diagram

$$\begin{array}{ccccccccc} 0 \to & X(s) & \to & X(s) \oplus A(s)^p & \to & M \otimes A(s) & \to 0 \\ & || & & \downarrow & & \downarrow & \\ 0 \to & X(s) & \to & X(s) & \to & 0 & \to 0 \end{array},$$

whose rows also are exact. Using the snake lemma, we see that δ is generically bijective. Further, we have a commutative square

$$\begin{array}{ccc} A(s)^p & \simeq & M \otimes A(s) \\ \downarrow & & || \\ X(s) \oplus A(s)^p & \to & M \otimes A(s) \end{array}.$$

Letting Φ denote the bottom arrow, we have $\begin{bmatrix} H(sI - F)^{-1} & I \end{bmatrix} = \delta^{-1}\Phi$. It follows that $\delta^{-1}\nu$ is equal to $H(sI - F)^{-1}G + J$. $\square$

Given a linear system Σ, the model constructed in the previous lemma is called the Fliess representation. The construction is functorial, and certainly preserves the property of regularity.

Proposition 1. *Taking the Fliess representation preserves the controllability and observability properties.*

Proof. Let $(X; F, G, H, J)$ be a linear system, and let $(M; \delta, \nu)$ be its Fliess representation. Applying the snake lemma to the first diagram in the proof above, we obtain an exact sequence

$$0 \to A[s]^p \to M \to X_F \to 0.$$

This sequence yields an isomorphism of $M/\delta A[s]^p$ onto X_F. Clearly the square

$$\begin{array}{ccc} A[s]^m & = & A[s]^m \\ \downarrow & & \downarrow \\ M/\delta A[s]^p & \simeq & X_F \end{array}.$$

is commutative. From this we get immediately that the cokernel of G_F is canonically isomorphic to that of $A[s]^m \to M/\delta A[s]^p$. Obviously the latter may be identified with $M/(\delta A[s]^p + \nu A[s]^m)$, and the statement about controllability follows.

Further, the exact sequence above fits into the commutative diagram

$$\begin{array}{ccccccc} 0 \to & A[s]^p & \to & M & \to & X_F & \to 0 \\ & || & & \downarrow & & \downarrow & \\ 0 \to & A[s]^p & \to & A(s)^p & \to & A(s)^p/A[s]^p & \to 0 \end{array}.$$

The donward arrow in the middle is the composition $M \to M \otimes A(s) \simeq A(s)^p$, and therefore its kernel is equal to $t(M)$. By the snake lemma, this is canonically isomorphic to the kernel of H_F, and the statement about observability follows. □

Suppose a Fliess model $(M; \delta, \nu)$ is given. We define the state space of it as

$$X = \{x \in M |\ \delta^{-1}x \in s^{-1}O^p\}.$$

Define linear maps $F : X \to X$, $H : A^m \to X$, $G : X \to A^p$ and $J : A^m \to A^p$ by the formulas

$$sx = Fx + \delta Hx \quad \text{and} \quad \nu(u) = Gu + \delta Ju.$$

These are well-defined. The linear system obtained is called the state-space representation of the model. The reader can notice that this definition is a direct generalization of the Fuhrmann classical definition given in [2].

Clearly taking the state-space representation is a functor.

Lemma 2. *Let $(M; \delta, \nu)$ be a Fliess model, and let $(X; F, G, H, J)$ be its state space representation. If $(M_1; \delta_1, \nu_1)$ is the Fliess representation of the latter, then there is a canonical isomorphism*

$$(M; \delta, \nu) \simeq (M_1; \delta_1, \nu_1).$$

Proof. Define $X[s] \to X[s] \oplus A[s]^p$ as $\begin{bmatrix} sI - F \\ -H \end{bmatrix}$, and let $X[s] \oplus A[s]^p \to M$ be the canonical homomorphism. We are going to show that the sequence

$$0 \to X[s] \to X[s] \oplus A[s]^p \to M \to 0$$

is exact. It is easy to check that the second arrow is injective and that its composition with the third one is zero. Next, the third arrow is surjective. Indeed, let x be an element of M. If h is the polynomial part of $\delta^{-1}(x/1)$, then clearly $x - \delta h \in X$, and the image of $(x - \delta h, h)$

is x. Thus, we need only to show that any element from $X[s] \oplus A[s]^p$ going to zero comes from $X[s]$ necessarily.

Let $(\sum_{i=0}^{l} x_i \otimes s^i, \sum_{i=0}^{l} a_i s^i)$ be any such element. By the hypothesis,

$$(x_0 + \delta a_0) + s(x_1 + \delta a_1) + \cdots + s^l(x_l + \delta a_l) = 0.$$

Put

$$\begin{array}{rcl} \xi_0 &=& -s^{-1}(x_0 + \delta a_0), \\ \xi_1 &=& -s^{-2}(x_0 + \delta a_0) - s^{-1}(x_1 + \delta a_1), \\ &\vdots& \\ \xi_{l-1} &=& -s^{-l}(x_0 + \delta a_0) - \cdots - s^{-1}(x_{l-1} + \delta a_{l-1}). \end{array}$$

Obviously we have

$$\begin{array}{rcl} s\xi_0 &=& -(x_0 + \delta a_0), \\ s\xi_1 &=& \xi_0 - (x_1 + \delta a_1), \\ &\vdots& \\ s\xi_{l-1} &=& \xi_{l-2} - (x_{l-1} + \delta a_{l-1}) \\ 0 &=& \xi_{l-1} - (x_l + \delta a_l). \end{array}$$

Using the relation $sx = Fx + \delta Hx$, we see that

$$\begin{array}{rclcrcl} -F\xi_0 &=& x_0, & & -H\xi_0 &=& a_0; \\ \xi_0 - F\xi_1 &=& x_1, & & -H\xi_1 &=& a_1; \\ &\vdots& & & &\vdots& \\ \xi_{l-2} - F\xi_{l-1} &=& x_{l-1}, & & -H\xi_{l-1} &=& a_{l-1}; \\ \xi_{l-1} &=& x_l, & & 0 &=& a_l. \end{array}$$

Hence,

$$\left[\begin{array}{c} sI - F \\ -H \end{array} \right] (\sum_{i=0}^{l-1} \xi_i \otimes s^i) \;=\; \sum_{i=0}^{l} \left[\begin{array}{c} x_i \otimes s^i \\ a_i \otimes s^i \end{array} \right].$$

It is now easy to complete the proof. Indeed, it follows from our exact sequence that there exists a unique isomorphism $\alpha : M \to M_1$ making the diagram

$$\begin{array}{ccccccc} 0 \to & X[s] & \to & X[s] \oplus A[s]^p & \to & M & \to 0 \\ & || & & || & & \downarrow & \\ 0 \to & X[s] & \to & X[s] \oplus A[s]^p & \to & M_1 & \to 0 \end{array}$$

commutative. Clearly, α is a transformation of models, and the proof is complete. □

Lemma 3. *Let $(X; F, G, H, J)$ be a linear system, and let $(M; \delta, \nu)$ be its Fliess representation. If $(X_1; F_1, G_1, H_1, J_1)$ is the state space representation of the latter, then there is a canonical isomorphism*

$$(X; F, G, H, J) \simeq (X_1; F_1, G_1, H_1, J_1).$$

Proof. If $x \in X$, then $H(sI-F)^{-1}x \in s^{-1}O^p$, and consequently the image of $(x,0)$ under the canonical epimorphism $X[s] \oplus A[s]^p \to M$ belongs to X_1. So, we have a canonical linear map $X \to X_1$. It is easily verified that this determines a transformation.

We claim that the linear map is bijective. Indeed, if an element $x \in X$ goes to zero, then $x = (sI-F)\xi$ for some $\xi \in X[s]$, and consequently $x = 0$. To show surjectivity take any element in X_1 represented, say, by $(x,a) \in X[s] \oplus A[s]^p$. Put $b = H(sI-F)^{-1}x + a$, which is contained in $s^{-1}O^p$. Because (x,a) and $(0,b)$ have the same image under the canonical map $\begin{bmatrix} H(sI-F)^{-1} & I \end{bmatrix} : X(s) \oplus A(s)^p \to A(s)^p$, we have

$$\begin{bmatrix} x \\ a-b \end{bmatrix} = \begin{bmatrix} sI-F \\ -H \end{bmatrix} \xi$$

for some $\xi \in X(s)$. We can write ξ as $\xi_1 + s^{-1}\xi_0 + s^{-2}\xi_2$, where $\xi_1 \in X[s]$, $\xi_0 \in X$ and $\xi_2 \in O(X)$. The equality above reduces to

$$x = s\xi_1 + \xi_0 + s^{-1}\xi_2 - F\xi_1 - s^{-1}F\xi_0 - s^{-2}F\xi_2 \quad \text{and} \quad a = b + H\xi_1 + s^{-1}H\xi_0 + s^{-2}H\xi_2.$$

We see that

$$x = \xi_0 + (sI-F)\xi_1 \quad \text{and} \quad a = H\xi_1.$$

It follows that our element is represented by $(\xi_0, 0)$ as well. □

Theorem 1. *The Fliess representation and state-space representation functors establish a canonical equivalence between the categories of linear systems and Fliess models.*

Proof. Follows from the previous two lemmas.

Corollary There is canonical one-to-one correspondence between isomorphism classes of minimal linear systems and transfer functions.

Proof. Follows from the theorem above and Example 1.

The theorem 3 of [4] and the theorem above tell us that the module-theoretic approaches of Kalman and Fliess are equivalent. For convenience of the reader, we briefly describe the direct link between these two approaches. We shall follow [3].

Let $(Q;\phi,\psi,J)$ be a Kalman model. Set

$$M = \{(x,v) \in Q \oplus A(s)^p |\ \psi(x) = \pi(v)\},$$

and define $\delta : A[s]^p \to M$ and $\nu : A[s]^m \to M$ by the formulas

$$\delta(v) = (0,v) \quad \text{and} \quad \nu(u) = (\phi(u), Tu),$$

where T is the transfer function of our model. It is easily seen that the canonical sequence $0 \to A[s]^p \to M \to Q \to 0$ is exact. This implies that M is finitely generated and that δ is generically bijective. So, $(M;\delta,\nu)$ is a Fliess model.

Suppose now a Fliess model $(M; \delta, \nu)$ is given. Set

$$Q = M/\delta A[s]^p,$$

and define $\phi : A[s]^m \to Q$ and $\psi : Q \to A(s)^p/A[s]^p$ by the formulas

$$\phi(u) = \nu(u)\text{mod}\delta A[s]^p \quad \text{and} \quad \psi(x\text{mod}\delta A[s]^p) = \delta^{-1}(x/1)\text{mod}A[s]^p.$$

Define J as the polynomial part of the transfer function of the model. Clearly $(Q; \phi, \psi, J)$ is a Kalman model.

It can be shown easily that the two constructions above are functorial and they determine an equivalence between Kalman models and Fliess models.

References

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