Approximation of optimal controls for semilinear parabolic PDE by solving Hamilton-Jacobi-Bellman equations.

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Abstract

This paper deals with a numerical approximation of optimal controls by solving a Hamilton-Jacobi-Bellman (HJB) equation, corresponding to control problems of parabolic PDE. The method is based on a model reduction, using POD (Proper Orthogonal Decomposition), and on the approximation of the HJB equation of the reduced problem by a finite difference scheme.

1 Introduction

We are interested in the computation of optimal controls for parabolic partial differential equation. For that, we develop the following method:

Using a POD basis (see K. Kunisch and S. Volkwein [4]), the problem is approximated by another control problem governed by an ODE of low order. The control of this reduced problem is calculated by solving the corresponding Hamilton-Jacobi-Bellman equation.

We apply this method to two different problems, for which we compare the known solution and its numerical approximation.

1. Consider the problem

$$\inf \left\{ J\left(u,y\right) \mid \left(u,y\right) \text{ satisfied } (1.1)\ ,\ u \in U\left[0,T\right] \right\}, \tag{P}$$

with the one-dimensional parabolic equation:

$$\begin{cases} y_t - y_{xx} = 0 \text{ in } \left]0,T\right[\times \Omega = \left]0,T\right[\times \left]0,L\right[, \\ \dfrac{\partial y}{\partial n} + \left|y\right|^3 y \Big|_{x=L} = \left(b\left(t\right) + u\left(t\right) - y\left(L,t\right)\right), \\ \dfrac{\partial y}{\partial n}\Big|_{x=0} = 0, \\ y(\cdot,0) = y_0\left(\cdot\right) \text{ in } \Omega, \end{cases} \tag{1.1}$$

[1]This work is a part of my PHD thesis, J.P. Raymond advisor, in collaboration with J.F. Bonnans and H.Zidani (Sydoco project), from INRIA Rocquencourt (France).

$$U[0,T]=\{u\in L^{\infty}(0,T;\mathbb{R})\mid u_a\le u(t)\le u_b \text{ for a.e. } t\in[0,T]\},$$

and

$$J(u,y)=\frac{\kappa}{2}\int_0^T u(t)^2\,dt+\frac{1}{2}\|y(T)-y_T\|^2_{L^2(\Omega)}+\int_0^T(-a(t)\,y(L,t)+d(t)\,u(t))\,dt. \tag{1.2}$$

2. The second problem is defined by replacing equation (1.1) by the following Burgers type equation:

$$\left\{\begin{array}{l} y_t-y_{xx}+yy_x=f+u \text{ in } \,]0,T[\times]0,L[\,, \\ y(0,t)=y(L,t)=0, \\ y(\cdot,0)=y_0(\cdot) \text{ in } \Omega. \end{array}\right. \tag{1.3}$$

2 Presentation and construction of the POD basis:

2.1 Finite Element discretization:

In these two examples, if $u\in L^{\infty}(0,T)$, the solution y of the state equation belongs to $L^2(0,T;H^1(\Omega))$ (and $y\in L^2(0,T;H^1_0(\Omega))$ for (1.3)) . For equation (1.1), we approximate $H^1(\Omega)$ by a $\mathbb{P}_1$-finite element method. We denote by V_h:

$$V_h=\left\{\varphi\in C\left(\overline{\Omega};R\right)\mid \varphi\big|_{[x_i,x_{i+1}]}\in\mathbb{P}_1\ \ i=1,N+1\right\}. \tag{2.4}$$

Let $y_h(t,x)=\sum_{i=1}^{N+1}y_i(t)\,\varphi_i(x)\in L^{\infty}(0,T;V_h)$, with $\varphi_i(x_j)=\delta_{ij}$. Setting

$$Y(t)=\left(\begin{array}{c} y_1(t)\\ \vdots \\ y_{N+1}(t)\end{array}\right),$$

we look for the solution Y of a nonlinear ODE of the form:

$$\left\{\begin{array}{l}\dot{Y}(t)=f_h(t,Y(t),u(t)),\\ Y(0)=Y_0.\end{array}\right.$$

We use an implicit (Euler) finite difference scheme for the time discretization. And we approximate the nonlinear solution by a Newton method. We obtain a matrix $(Y_{ij})_{i,j}$ (where $(Y_{ij})\simeq y(x_i,t_j)$). For equation (1.3), we calculate an approximate solution of the equation by using $N-1$ components (φ_1 and φ_{N+1} are identically zero).

2.2 Presentation of the Reduced model, POD:

Goal: For an element $y_h\in L^2(0,T;V_h)$, where $V_h\subset L^2(\Omega)$ is a $n-$dimensional subspace, we want to minimize the gap

$$\|y_h(\cdot,\cdot)-z(\cdot,\cdot)\|_{L^2(0,T;L^2(\Omega))},$$

where the function z is looked of the form:

$$z(t,\cdot)=\sum_{i=1}^{l} z_i(t)\Psi^i(\cdot), \tag{2.5}$$

$l << n$, and

$$\left\{\Psi^i\right\}_{i=1}^{l} \text{ is an orthonormal basis of } L^2(\Omega) \text{ satisfying } : \Psi^i=\sum_{j=1}^{n}\Psi_{ji}\varphi_i(\cdot). \tag{2.6}$$

Construction: We take the set of N_T+1 vectors $\{y_h(\cdot,t_j)\}_{j=1}^{N_T+1}$ of $L^2(\Omega)$:

$$y_h(\cdot,t_j)=\sum_{i=1}^{n} Y_{ij}\varphi_i(\cdot). \tag{2.7}$$

The functions Ψ^i are determined as the solution of the optimization problem:

$$\min_{\substack{\left\{\Psi^i\right\}_{i=1}^{l}\\ \left\langle\Psi^i,\Psi^j\right\rangle=\delta_{ij}}} \sum_{j=1}^{N_T+1}\alpha_j\left\|y_h(\cdot,t_j)-\sum_{i=1}^{l}\left\langle y_h(\cdot,t_j),\Psi^i\right\rangle\Psi^i\right\|_{L^2(\Omega)}^2, \tag{2.8}$$

where

$$\alpha_j=\Delta t,\ j=2,..,N_T,\ \alpha_j=\frac{\Delta t}{2},\ j=1 \text{ and } N_T+1. \tag{2.9}$$

Remark 2.1. *The reals α_j are chosen as in* (2.9) *because*

$$\mathcal{I}_n(y)=\sum_{j=1}^{N_T+1}\alpha_j\left\|y_h(\cdot,t_j)-\sum_{i=1}^{l}\left\langle y_h(\cdot,t_j),\Psi^i\right\rangle\Psi^i\right\|_{L^2(\Omega)}^2$$

is the trapeze method approximation of the integral:

$$\mathcal{I}(y)=\int_0^T\left\|y_h(\cdot,t)-\sum_{i=1}^{l}\left\langle y_h(\cdot,t),\Psi^i\right\rangle\Psi^i\right\|_{L^2(\Omega)}^2 dt.$$

And for all $y\in C\left([0,T];L^2(\Omega)\right)$ we have:

$$\mathcal{I}_n(y)\rightarrow\mathcal{I}(y) \text{ as } n\rightarrow\infty.$$

So, we denote by Θ and B the positive definite matrices such that

$$\Theta_{ji}=0,\ \text{if } i\neq j,\ \Theta_{jj}=\sqrt{\alpha_j},\ j=1,..,N_T+1,\ \text{and } B_{ij}=\int_0^L\varphi_i\varphi_j dx.$$

We deduce from [4], that the matrix Ψ (given in (2.6)) and the vectors Ψ^i, are determined by the algorithm:

1. Set: $\tilde{Y} = B^{\frac{1}{2}} Y \Theta$
2. $U \Sigma V^T = SVD(\tilde{Y})$
3. $\bar{U}_{ij} = U_{ij}$ for $1 \leq i \leq n$, and $1 \leq j \leq l$
4. $\Psi = B^{-\frac{1}{2}} \bar{U}$,

where SVD is the Singular Value Decomposition.

The number l of elements chosen for the reduced model, depends on the ratio:

$$\varepsilon(l) = \frac{\sum_{i=1}^{l} \sigma_i^2}{\sum_{i=1}^{rg(\tilde{Y})} \sigma_i^2} \tag{2.10}$$

where the $(\sigma_i)_{1 \leq i \leq rg(\tilde{Y})}$ are the singular values of $\tilde{Y}$.

The error estimation between an element $y_h(\cdot, t_j)$ and its projection on the POD basis is given by:

$$Err(l) = \left(\sum_{j=1}^{N_T+1} \alpha_j \left\| y_h(\cdot, t_j) - \sum_{i=1}^{l} \left\langle y_h(t_j), \Psi^i \right\rangle \Psi^i \right\|_{L^2(\Omega)}^2 \right)^{1/2} = \left(\sum_{i=l+1}^{rg(\tilde{Y})} \sigma_i^2 \right)^{1/2} \tag{2.11}$$

The proof of this result is adapted from [5].

2.3 Application.

For the two problems let us give some results.

1. For (1.1): With $u \equiv 0$, $N = 30$ and $N_T = 38$, we have:

l	1	2	3	4
$\varepsilon(l)$	0.9895643	0.9999941	1.0^-	1.0^-
$Err(l)$	0.0496332	0.0011831	.0000419	.0000018 .

2. For (1.3), with u, N and N_T as above:

l	1	2	3	4	5
$\varepsilon(l)$	0.8078	0.9528	0.9849	0.9944	0.9979
$err(l)$	0.2549510	.1256981	.0707107	.0424264	.0264575

Remark 2.2. *In the second case, we need 3 or 4 elements of the POD basis to have the same precision as for (1.1) with only $l = 1$ pod element.*

Now, we are going to define the reduced model, and the reduced optimal control problem. So, first, we look for the initial value condition z_0 solution of

$$\min_{z_0=\sum_{i=1}^{l} z_{0i}\Psi^i} \left\| y_0^h - z_0 \right\|_{L^2(\Omega)}^2 .$$

In fact z_0 is the orthogonal projection of y_0^h on the $l-$dimensional subspace spanned by the $\Psi^i, i = 1, .., l$. So we have

$$\begin{aligned} z_{0j} &= \left\langle y_0^h, \Psi^j \right\rangle_{L^2(\Omega)} \\ &= \sum_{i,k=1}^{N+1} \Psi_{ij} y_0^h(x_k) \int_\Omega \varphi_k \varphi_i \\ &= \left(Y_0^T B \Psi \right)_j \end{aligned}$$

where $Y_0 = (y_0(x_i))_{i=1}^{N+1}$. Now, as for the Finite Element method, we obtain in the two cases (1.1), (1.3) an ODE of the form:

$$\left\{ \begin{array}{l} \dot{z}(t) = f_{pod}(t, u(t), z(t)) \\ z(0) = z_0, \end{array} \right. \tag{2.12}$$

with

$$z(t, x) = \sum_{i=1}^{l} z_i(t) \Psi^i(x) .$$

We solve (2.12) by an implicit finite difference scheme, and we obtain POD solutions dependent of l, of the initial value condition, and of the command u. We have plotted error estimations in Figures 1 and 2.

For $l = 1$ *pod*, we see the maximal error on the Figures 1, and 2 are respectively 0.12 and 0.247.
For $l = 2$ *pod*, and $u = 1/2$ the maximal error is less than 5.66×10^{-3}.
For $l = 2$ *pod*, and $u = 0$ the maximal error is less than 2.10×10^{-4}.

Remark 2.3. *For the first equation, the kind of values for $\varepsilon(l)$ is very characteristic of this type of equation (the heat as) and the regularity used. The solution is very smooth, and its variations are small. We can see that it is very different in the second one.*

2.4 Approximation of the control problem:

Now we look at the control problem

$$\inf \left\{ J_{t,z_0}^{pod}(u, z) \mid (u, z) \text{ satisfied (2.12) on }]t, T] , u \in U[t, T] \right\},$$

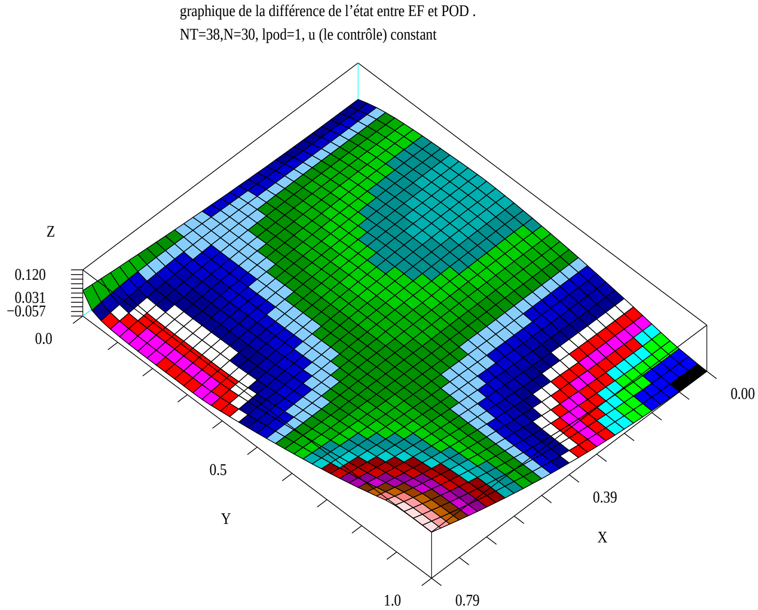

Figure 1: Difference between the FE state and the POD state, with $l = 1$, $u = 1/2$.

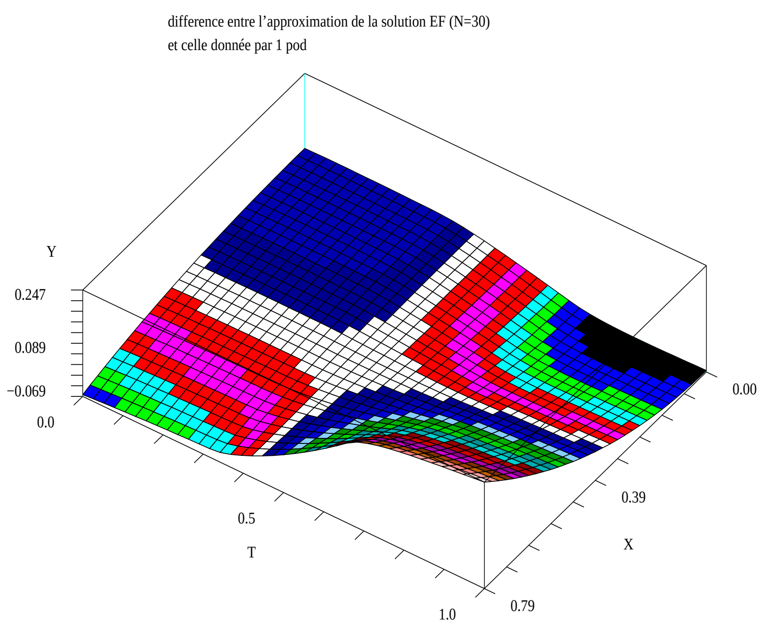

Figure 2: Difference between the FE state and the POD state, with $l = 1$, $u = 0$.

with the new initial condition at t time:

$$z(t) = z_0, \tag{2.13}$$

where:

$$\begin{aligned} J^{pod}(u,z) &= \int_0^T l(s, z(s), u(s))\, ds + \varphi(z(T)) && (2.14) \\ l(s,z,u) &= \frac{\kappa}{2} u^2 + a_u(s)\, u - a_y(s) \sum_{i=1}^{l} \Psi_{N+1,i} z_i \\ \varphi(z) &= \frac{1}{2} \|z - z_T\|^2_{L^2(\Omega)} = \frac{1}{2} \sum_{i=1}^{l} (z_i - z_{T_i})^2 . && (2.15) \end{aligned}$$

The value function is defined by:

$$v^{pod}(t, z_0) = \inf_{u \in U[t,T]} J^{pod}_{t,z_0}(u, z).$$

It is known (see [3] or [2] for example) that v^{pod} is the viscosity solution of the Hamilton-Jacobi-Bellman equation:

$$\begin{cases} -v_t(t, z_0) + \mathcal{H}(t, z_0, D_y v(t, z_0)) = 0, \\ v(T, z_0) = \varphi(z_0), \end{cases} \tag{2.16}$$

where $\mathcal{H}$ is the Hamiltonian function of the problem:

$$\mathcal{H}(t, z_0, p) = \sup_{u \in U} \{-l(t, z_0, u) - p \cdot f_{pod}(t, z_0, u)\}.$$

3 Numerical scheme:

The method we are going to present is based on an explicit finite difference scheme, where the control $u \in U$ takes a finite number of values. The convergence of this scheme follows from a monotonicity and a stability condition (see [1] or [6]). This kind of scheme has been studied in [8], where it is applied it to several simple examples.

3.1 Discretisation of the Hamilton-Jacobi-Bellman equation:

We are going to present the scheme to discretize (2.16) in dimension l.

Set $w(t,z) = v^{pod}(T-t, z)$, and we consider the initial value problem:

$$\begin{cases} w_t(t,z) + \sup_{u \in U} \left[-\sum_{k=1}^{l} f^k_{pod}(T-t, z, u) \cdot \partial^k w - l(T-t, z, u) \right] = 0, \\ w(0,z) = \varphi(z), \end{cases} \tag{3.17}$$

where $z=(z_1,..,z_l)\in\mathbb{R}^l$, and $(t,z)\in\,]0,T]\times\prod_{k=1}^{l}[a_k,b_k]$. We denote by

$$\begin{aligned}
i &= (i_1,i_2,...,i_l)\,,ik^+=(i_1,i_2,..,i_{k-1},i_k+1,..,i_l)\,,\\
ik^- &= (i_1,i_2,..,i_{k-1},i_k-1,..,i_l)\,,\\
z_{k,i_k} &= a_k+i_k\Delta z_k, i_k=0....M_k,\ \text{with } \Delta z_k=(b_k-a_k)/M_k,\\
t_n &= (n-1)\Delta t,\quad n=1...NT+1,\ \text{with } \Delta t=T/NT.
\end{aligned}$$

Algorithm 3.1.

% Initialisation of $\{w_i^1\}_{i_k=0}^{M_k}\quad k=1,...,l$

<u>*For*</u> $n=1:NT$

<u>*For*</u> $i_k=n:M_k-n,\ k=1,...,l$

$\mathcal{H}_i^n(u)=\left[-\sum_{k=1}^{l}\left(f_{k,i}^{n^+}(u)\cdot D_{ik}^+w^n+f_{k,i}^{n^-}(u)\cdot D_{ik}^-w^n\right)-l\left(T-t_n,z_i,u\right)\right]$

$u_i^n=\arg\sup_{u\in U_k}\mathcal{H}_i^n(u)$

$w_i^{n+1}=w_i^n-\Delta t\mathcal{H}_i^n(u_i^n)$

<u>*end*</u>*;*

<u>*end*</u>*;*

where

$$\begin{aligned}
f_{k,i}^{n^+}(u) &= \max\left(f^k\left(T-t_n,z_i,u\right),0\right),\ f_{k,i}^{n^-}(u)=\min\left(f^k\left(T-t_n,z_i,u\right),0\right) &(3.18)\\
D_{ik}^+w &= \frac{w_{ik^+}-w_i}{\Delta z_k},\ D_{ik}^-w=\frac{w_i-w_{ik^-}}{\Delta z_k}, &(3.19)
\end{aligned}$$

and U_k is a finite subset of U.

Remark 3.1. *The lines (3.18) and (3.19), mean that we use decentered finite differences: we decentrate at left when $f^k\left(T-t_n,z_i,u\right)$ is negative, and at the right if it is positive.*

Remark 3.2. *We can see that even if at the beginning the computational region a hyper-pyramidal region, it is needed to compute all the values w_i^n.*

3.1.1 Monotony of the scheme :

If we write w_i^{n+1} in function of $w_{i-1}^n,w_i^n,w_{i+1}^n$, we have:

$$\begin{aligned}
w_i^{n+1} &= w_i^n-\Delta t\left[-\sum_{k=1}^{l}\left(f_{k,i}^{n^+}(u_i^n)\cdot D_{ik}^+w^n+f_{k,i}^{n^-}(u_i^n)\cdot D_{ik}^-w^n\right)-l\left(T-t_n,z_i,u_i^n\right)\right]\\
&= \left[1-\Delta t\sum_{k=1}^{l}\frac{\left(f_{k,i}^{n^+}(u_i^n)-f_{k,i}^{n^-}(u_i^n)\right)}{\Delta z_k}\right]w_i^n+\Delta t\sum_{k=1}^{l}\frac{f_{k,i}^{n^+}(u_i^n)\,w_{ik^+}^n-f_{k,i}^{n^-}(u_i^n)\,w_{ik^-}^n}{\Delta z_k}\\
&\quad+\Delta t\ l\left(T-t_n,z_i,u_i^n\right). &(3.20)
\end{aligned}$$

Besides:

- $f^{+} - f^{-} = |f|$, and

$$1 - \Delta t \sum_{k=1}^{l} \frac{\left(f_{k,i}^{n^{+}}\left(u_i^n\right) - f_{k,i}^{n^{-}}\left(u_i^n\right)\right)}{\Delta z_k} = 1 - \Delta t \sum_{k=1}^{l} \frac{\left|f_{k,i}^{n}\left(u_i^n\right)\right|}{\Delta z_k}. \tag{3.21}$$

We choose Δt satisfying the (CFL) condition:

$$\sum_{k=1}^{l} \frac{\left\|f^k\right\|_{\infty}}{\Delta z_k} \Delta t \leq 1, \tag{CFL}$$

where $\left\|f^k\right\|_{\infty} = \max_{(t,x,u)\in[0,T]\times\prod_{k=1}^{l}[a_k,b_k]\times U} \left|f_{pod}^k(t,x,u)\right|$. Therefore:

$$(3.21) \geq 1 - \Delta t \sum_{k=1}^{l} \frac{\left\|f^k\right\|_{\infty}}{\Delta z_k} \geq 0.$$

- Since $f^{+} \geq 0$ and $-f^{-} \geq 0$,

reporting this inequality in (3.20), we see that w_i^{n+1} is a non-decreasing monotone function of w_{i-1}^n, w_i^n and w_{i+1}^n.

3.1.2 Stability of the scheme:

As shown in [1], the CFL condition implies the stability of the scheme. It is shown in [7] that it is sufficient to verify the stability condition for $l = 0$. If we take $l = 0$ in (3.20) we have:

$$\begin{aligned}
\left|w_i^{n+1}\right| &= \left|\left[1 - \Delta t \sum_{k=1}^{l} \frac{\left(f_{k,i}^{n^{+}}\left(u_i^n\right) - f_{k,i}^{n^{-}}\left(u_i^n\right)\right)}{\Delta z_k}\right] w_i^n + \Delta t \sum_{k=1}^{l} \frac{f_{k,i}^{n^{+}}\left(u_i^n\right) w_{ik^{+}}^n - f_{k,i}^{n^{-}}\left(u_i^n\right) w_{ik^{-}}^n}{\Delta z_k}\right| \\
&\leq \left|1 - \Delta t \sum_{k=1}^{l} \frac{\left(f_{k,i}^{n^{+}}\left(u_i^n\right) - f_{k,i}^{n^{-}}\left(u_i^n\right)\right)}{\Delta z_k}\right| \left|w_i^n\right| + \Delta t \left|\sum_{k=1}^{l} \frac{f_{k,i}^{n^{+}}\left(u_i^n\right) w_{ik^{+}}^n - f_{k,i}^{n^{-}}\left(u_i^n\right) w_{ik^{-}}^n}{\Delta z_k}\right| \\
&\leq \left|1 - \Delta t \sum_{k=1}^{l} \frac{\left|f_{k,i}^{n}\left(u_i^n\right)\right|}{\Delta z_k}\right| \sup_i \left|w_i^n\right| + \Delta t \sup_i \left|w_i^n\right| \sum_{k=1}^{l} \frac{\left|f_{k,i}^{n^{+}}\left(u_i^n\right)\right| + \left|f_{k,i}^{n^{-}}\left(u_i^n\right)\right|}{\Delta z_k} \\
&= \left(\left|1 - \Delta t \sum_{k=1}^{l} \frac{\left|f_{k,i}^{n}\left(u_i^n\right)\right|}{\Delta z_k}\right| + \Delta t \sum_{k=1}^{l} \frac{\left|f_{k,i}^{n}\left(u_i^n\right)\right|}{\Delta z_k}\right) \sup_i \left|w_i^n\right|. \qquad (3.22)
\end{aligned}$$

From the (CFL) condition it follows that

$$\left|1 - \Delta t \sum_{k=1}^{l} \frac{\left|f_{k,i}^{n}\left(u_i^n\right)\right|}{\Delta z_k}\right| = 1 - \Delta t \sum_{k=1}^{l} \frac{\left|f_{k,i}^{n}\left(u_i^n\right)\right|}{\Delta z_k},$$

and (3.22) becomes

$$\left|w_i^{n+1}\right| \leq \sup_i \left|w_i^n\right|,$$

that is

$$\left\|w^{n+1}\right\|_\infty \leq \left\|w^n\right\|_\infty \leq \left\|w^0\right\|_\infty.$$

Q.E.D.

To validate our algorithm, we have applied the scheme in some simple examples in dimension one and two. Now, we are going to apply this scheme in our reduced problem.

3.2 Application of the scheme on the reduced model:

We denote

$$v^{pod}(t,z) = \inf_{u \in U} J_{t,z}^{pod}(u, y_{t,z})$$

where $J_{t,z}^{pod}$ is given in (2.14). We propose the following approach:

1. Determine a set of trust region within live the coordinates of the POD solution: $y_{t,z}$,
2. Apply the finite difference scheme,
3. Rebuild the optimal control.

Remark 3.3. *For the moment we have not tested all of that for the Burger's equation. So, the end of the paper is devoted to the first example.*

3.2.1 Set of trust region for the POD solution for the first problem:

We can show that the first coordinate obeys $-0.802 < z_{0_1} \leq 0$,, and by empirical estimation, $z_2 \in [-0.02, 0.05]$. Other coordinates belongs to smaller intervals.

Proof. Let us shown that $|z_1| \leq 0.802$. We have first:

$$\begin{aligned} z_{0_j} &= \left(Y_0^T B \Psi^j\right) \\ &= \left\langle Y_0, B\Psi^j \right\rangle_{\mathbb{R}^{N+1}} \\ &= \left\langle B^{\frac{1}{2}} Y_0, B^{\frac{1}{2}} \Psi^j \right\rangle_{\mathbb{R}^{N+1}} \end{aligned}$$

so:

$$\left|z_{0_j}\right| \leq \left\|B^{\frac{1}{2}} Y_0\right\|_{\mathbb{R}^{N+1}} \left\|B^{\frac{1}{2}} \Psi^j\right\|_{\mathbb{R}^{N+1}},$$

and else

$$\left\|B^{\frac{1}{2}} \Psi^j\right\|_{\mathbb{R}^{N+1}} = \left\|U^j\right\|_{\mathbb{R}^{N+1}} = 1 : U^j \text{ unitary vector from the SVD},$$

$$\left|z_{0_j}\right| \leq \left\|B^{\frac{1}{2}} Y_0\right\|_{\mathbb{R}^{N+1}}.$$

It is sufficient to numerically compute this value to have the result on the first coordinate, since

$$\left\|B^{\frac{1}{2}} Y_0\right\|_{\mathbb{R}^{N+1}} \leq 0.802.$$

We can even choose the sign of z_{0_j}: it is enough to change Ψ^j in $-\Psi^j$, or (this is the same) to change the column vectors sign u_j in $-u_j$ in matrix U (from the SVD since $\Psi = B^{-\frac{1}{2}}U$). U keeps its unitary properties.

□

3.2.2 Rebuilding methods of the optimal control:

There are several possible choices: the first one discretize the minimisation of the Hamiltonian, because the optimal u for the problem (1.1) (or (1.3)) is the same as the one who minimise the Hamiltonian. This method avoid stocking a big matrix $\overline{u}_j^n$, full of optimal values of the control for each point of the grid. The second one is based on the interpolation of these stocked values $\overline{u}_j^n$, and the third one is by taking closest values of the control on the grid. We will give the algorithm, only in the first case.

1. **Minimisation of the Hamiltonian :** after having determinated the value function approximation: $\widehat{v}$, we do another loop on the time-discretisation to determine the optimal control. Besides the reconstruction of the POD state can be done by an implicit or an explicit scheme:

   For the explicit one:

$$\left\{\begin{array}{l} u^{n+1} = \arg\min_{u\in U}\left[l(Z(t_n),u) + f_{pod}\left(Z(t_n),u,t_n\right)\cdot D\widehat{v}\left(t_n, Z(t_n)\right)\right] \\ Z(t_{n+1}) = Z(t_n) + \Delta t. f_{pod}\left(Z(t_n), u^{n+1}, t_n\right) \end{array}\right.$$

   Algorithm 3.2.

   $Z^1 = Z_0$ *% the initial value condition*

   For $n = 1 : NT$

   $\tilde{F} = f_{pod}(Z^n, U, t_n)$

   $j = \max(find(X1 \leq Z^n))$ *% the sufficies* j_i *s.t* $Z^n \in \prod_{i=1}^{l}\left[X1_{j_i}, X1_{j+1_i}\right]$

   $H = l(Z^n, U, T - t_n) + \sum_{i=1}^{l}\left(Val^n(j_{i+}) - Val^n(j_i)\right)\cdot \tilde{F}_i/\Delta z_i$

   $u^{n+1} = \arg\min_U(H)$

   $Z^{n+1} = Z^n + \Delta t. f_{pod}\left(Z^n, u^{n+1}, t_n\right)$

   end

   For the implicit one:

$$\left\{\begin{array}{l} Z(t_{n+1}) = Z(t_n) + \Delta t. f_{pod}\left(Z(t_{n+1}), u^{n+1}, t_{n+1}\right), \\ u^{n+1} = \arg\min_{u\in U}\left[l(Z(t_{n+1}),u) + f_{pod}\left(Z(t_{n+1}),u,t_{n+1}\right)\cdot D\widehat{v}\left(t_{n+1}, Z(t_{n+1})\right)\right]. \end{array}\right.$$

In the algorithm, we must do another Newton method; besides to find a first value of u we use the Algorithm 3.2:

Algorithm 3.3.

$Z^1 = Z_0;$ % *The first* u :
$\tilde{F} = f_{pod}(Z^n, U, t_n);$
$j = \max(find(X1 \leq Z^1))$
$H = l(Z^1, U, T) + \sum_{i=1}^{l} (Val^1(j_{i+}) - Val^1(j_i)) \cdot \tilde{F}_i / \Delta z_i$
$u = \arg\min_U(H)$
For $n = 1 : NT$
  $Z_2 = Z^n;$
  while $\|F(Z_2, u, Z^n)\| > \varepsilon$ % F *the newton function.*
    $\tilde{F} = f_{pod}(Z_2, U, t_{n+1}); j = \max(find(X1 \leq Z_2));$
    $H = l(Z_2, U, T - t_{n+1}) + \sum_{i=1}^{l} (Val^n(j_{i+}) - Val^n(j_i)) \cdot \tilde{F}_i / \Delta z_i;$
    $u = \arg\min_U(H)$
    computation of DF; $d = -F(Z_2, u, Z^n)/DF;$
    $Z_2 = d + Z_2;$
  end (while)
  $u\_opt^{n+1} = u$

2. **Interpolation:** We are going to present it only when $l = 1$ pod. For the implicit scheme we have:

$$\begin{cases} Z(t_{n+1}) = Z(t_n) + \Delta t. f_{pod}(Z(t_{n+1}), u^{n+1}, t_{n+1}) \\ \text{find } \lambda \text{ s.t. } Z(t_{n+1}) = \lambda \overline{Z}_j + (1-\lambda)\overline{Z}_{j+1} \\ u^{n+1} = \lambda \overline{u}_j^{n+1} + (1-\lambda)\overline{u}_{j+1}^{n+1}, \end{cases}$$

where $\overline{u}$ is the matrix with all the optimal values of the controls, and $\overline{Z}$ the vector grid of the space pod coordinates.

3. **Closest values:** For the implicit scheme we have:

$$\begin{cases} Z(t_{n+1}) = Z(t_n) + \Delta t. f_{pod}(Z(t_{n+1}), u^{n+1}, t_{n+1}) \\ \text{find } j \text{ s.t. } \left|Z(t_{n+1}) - \overline{Z}_j\right| \leq \left(\left|Z(t_{n+1}) - \overline{Z}_i\right|\right)_i \text{ for all } i \\ u^{n+1} = \overline{u}_j^{n+1}. \end{cases}$$

3.3 Numerical results and graphics:

To minimize the dependence and to improve the results, we applied several times the algorithm in injecting the found sub-optimal control. After 3 or 4 times, it is stabilized and we observe that it is the one we were looking for. For all the Figures presented (3, 4, 5), we use only one POD. For the first equation, we can show different graphics. With the interpolation method, we can see Figures 3 and 4 that after 3 iterations of the global algorithm, the

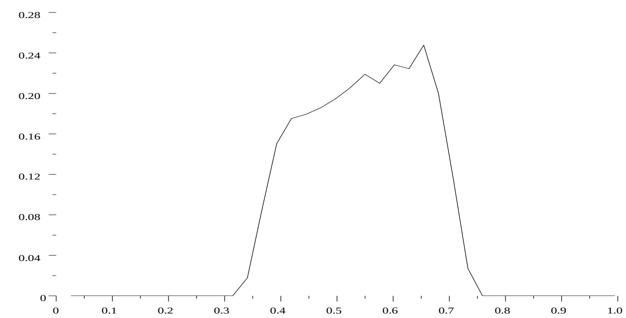

Figure 3: Error estimation when $u = 0$ with only one global iteration, interpolation

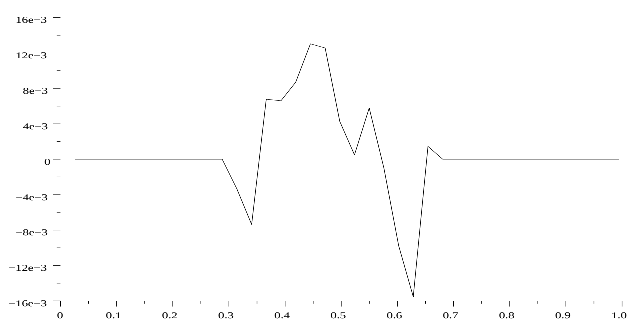

Figure 4: Error estimation when $u = 0$ with 3 global iteration, interpolation

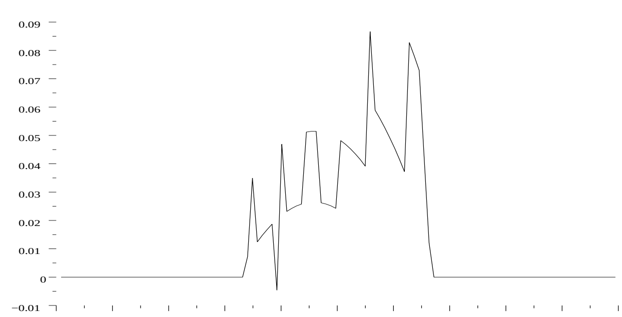

Figure 5: Error estimation when $u = 1/2$ with 1 global iteration, minimisation of the hamiltonian

maximal error is about 0.016. With the minimization of the Hamiltonian, we see Figure 5, that the maximal error is about 0.09 with only one iteration of the global algorithm, when we take $u = 1/2$ at the beginning of the program. In this example if we use more than one POD element, the optimal control found is very bad. It can be explained by the size of Δz_k. The CFL condition implies that Δz_k must be really bigger than Δt. And the size of the second trust region is about 0.07, whereas $\Delta z_2 \approx 0.12$ if we want to have a reasonnable time computation, indeed, we must add 2 N_T steps in each direction of space. And bigger is l, higher is the number of steps. So, to find a nice number l of pod elements, we need to look carefully all these sizes.

Remark 3.4. *In the first problem, if we bring a small perturbation, in front of the term* $|y|^3 y$*, and if we change the set of admissible controls, 2 pod elements are needed to approximate this new problem. For the moment the results found are not so nice.*

4 Conclusion:

In the case of example 1, we have obtained promizing results with only one element of the POD basis. The dependence of the basis with respect to the initial control, is not so true. We want to improve this method for applying it to the Burger's example, and later in two dimensional problems.

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