Further Results on Interconnection and Elimination for Delay-Differential Systems

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Abstract

In this short note we will consider a problem concerning the interconnection of systems from the behavioral point of view. The specific class of systems under consideration will be smooth time-invariant delay-differential systems with commensurate delays. We will give a characterization for the achievability of a given subsystem via regular interconnection from the overall system. This also leads to a criterion as to which systems can be achieved via regular interconnection followed by an elimination of the (then) latent variables. The criteria can be checked directly by some matrix computations.

1 Introduction

In the behavioral approach a system (more precisely, its behavior) is given as the set of all its trajectories, see [5]. In this note we will restrict to linear systems which are described by a finite set of equations. To be precise, let $\mathcal{L} \subseteq \mathbb{C}^{\mathbb{R}^n}$ be a $\mathbb{C}$-vector space of functions and let $\mathcal{H} \subseteq \text{End}_{\mathbb{C}}(\mathcal{L})$ be a ring of linear operators acting on $\mathcal{L}$. Then, for us a system with q variables is a space

$$\mathcal{B} := \ker_{\mathcal{L}} R := \{w \in \mathcal{L}^q \mid Rw = 0\} \tag{1.1}$$

where $R \in \mathcal{H}^{p\times q}$ for some $p \in \mathbb{N}$ is an operator matrix, called a kernel representation of the behavior $\mathcal{B}$. The q coordinates of w are called the external variables of the system. They comprise the inputs and outputs of the system (for the behavioral definition of these notions see [5]). The quite general definition (1.1) covers for instance the following classes of (time-invariant) systems:

(a) linear systems described by ODEs, where $\mathcal{H} = \mathbb{R}[\frac{d}{dt}]$ is the ring of linear ordinary differential operators acting on, say, $\mathcal{L} = \mathcal{C}^\infty(\mathbb{R}, \mathbb{C})$,

(b) linear multidimensional systems described by partial differential equations, where the ring $\mathcal{H} = \mathbb{R}[\partial_1, \ldots, \partial_n]$ acts on, say, $\mathcal{C}^\infty(\mathbb{R}^n, \mathbb{C})$,

(c) linear systems described by delay-differential equations, which will be specified in Section 3.

In the general situation above, the space $\mathcal{L}$ carries the structure of a left $\mathcal{H}$-module. Developing a behavioral theory for a certain class of systems (like (a), (b), or (c)) highly depends on the ring structure (resp. module structure) of $\mathcal{H}$ (resp. $\mathcal{L}$). While for instance the behavioral theory of multidimensional systems can make use of the powerful machine of commutative algebra [4, 8], this is not the case for delay-differential systems due to a different ring- or module structure (it depends on the setting which one is responsible). The differences have been pointed out in [3].

In the next section we will describe the notion of (regular) interconnection for general systems of the type (1.1). Interconnection usually leads to so-called latent variables in the new system, which one wants to eliminate from the systems description. We will illustrate this by an example. This consideration raises the question as to which systems can be obtained by interconnection and elimination from a given system. Finally, in the last section we will introduce a class of systems described by delay-differential equations and investigate the question posed above for these systems.

2 Interconnection of Systems

One of the most important concepts of control theory is that of feedback control. The first step towards this direction is that of interconnecting two systems, the plant and the controller. This comes very naturally in the behavioral setting as it can be expressed without resorting to the notions of inputs and outputs. The following definition is taken from [7, p. 332], where it has been introduced for linear time-invariant systems described by ODEs.

Definition 2.1
Let $\mathcal{L} \subseteq \mathbb{C}^{\mathbb{R}^n}$ be a $\mathbb{C}$-vector space of functions and let $\mathcal{H} \subseteq \mathrm{End}_{\mathbb{C}}(\mathcal{L})$ be a commutative ring of linear operators on $\mathcal{L}$.
Given two systems $\mathcal{B}_1, \mathcal{B}_2 \subseteq \mathcal{L}^q$. The interconnection of these systems is defined as the system $\mathcal{B}_1 \cap \mathcal{B}_2$, that is, it consists of all trajectories which satisfy both sets of dynamical equations. In particular, the interconnection of two systems $\mathcal{B}_1 = \ker_{\mathcal{L}} R_1$ and $\mathcal{B}_2 = \ker_{\mathcal{L}} R_2$ having kernel representations $R_i \in \mathcal{H}^{p_i \times q}$ is given by $\mathcal{B}_1 \cap \mathcal{B}_2 = \ker_{\mathcal{L}} \begin{bmatrix} R_1 \\ R_2 \end{bmatrix}$. The interconnection is called regular if $\mathrm{rk} \begin{bmatrix} R_1 \\ R_2 \end{bmatrix} = \mathrm{rk}\, R_1 + \mathrm{rk}\, R_2$.
For systems $\mathcal{B}_0, \mathcal{B}_1 \subseteq \mathcal{L}^q$ we say that $\mathcal{B}_0$ is achievable via regular interconnection from $\mathcal{B}_1$ if there exists a system $\mathcal{B}_2$ such that $\mathcal{B}_0 = \mathcal{B}_1 \cap \mathcal{B}_2$ and this interconnection is regular.

The notion of a regular interconnection is based on the following consideration. For many system classes, in particular the three classes (a)–(c) described above, the rank of a kernel representation is an invariant of the system. Furthermore, the number q of external variables reduced by the rank of a kernel representation represents the number of input variables of the system, where an input variable is an external variable which can be set freely (see [5]). If we think of the interconnecting system as a controller, it should be clear that each linear independent equation of the controller should put a restriction onto one input channel, otherwise the controller would be inefficient. This leads exactly to the rank condition above.

We illustrate this notion by an example of linear systems described by ODEs.

Example 2.2
Let $\mathcal{L} = \mathcal{C}^\infty(\mathbb{R}, \mathbb{C})$ and $\mathbb{R}[\frac{d}{dt}]$ acting on $\mathcal{L}$. Given two systems

$$\mathcal{B}_i = \left\{ \binom{u_i}{y_i} \in \mathcal{L}^{p+m} \,\middle|\, P_i u_i + Q_i y_i = 0 \right\},$$

where $q = p + m$ and $[P_1, Q_1] \in \mathbb{R}[\frac{d}{dt}]^{p\times(m+p)}$ and $[P_2, Q_2] \in \mathbb{R}[\frac{d}{dt}]^{m\times(p+m)}$. If $\det Q_1 \neq 0 \neq \det Q_2$, the two systems are input/output systems in the sense of [5, Def. 3.3.1] with input u_i and output y_i, see [5, Cor. 3.3.14]. The classical feedback-interconnection given by $u := u_1 - y_2$, $y_1 = u_2 =: y$ is described by the system

$$\ker_{\mathcal{L}} \begin{bmatrix} I & 0 & -I & I \\ 0 & Q_1 & P_1 & 0 \\ 0 & P_2 & 0 & Q_2 \end{bmatrix} \tag{2.1}$$

for the variables (u, y, u_1, y_2). It can be regarded as the interconnection of two suitably defined systems.

From the very definition it is obvious that the interconnection of two systems each having q variables still has q external variables. It is simply a subsystem of either of its components. But this means that, as in the example above, even the connected variables are still contained in the systems description, see (2.1). Usually one wants eliminate these connected variable (the so-called latent variables) from the systems description in order to get a kernel representation for the new system in terms of the relevant variables (the so-called manifest variables). For the example above this can easily be achieved as follows.

Example 2.3
Consider the system (2.1) in the situation of Example 2.2. If one is interested in the new external variables u and y only, one has to eliminate the latent variables u_1 and y_2 by taking the projection

$$\mathcal{B} := \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \end{bmatrix} \left(\ker_{\mathcal{L}} \begin{bmatrix} I & 0 & -I & I \\ 0 & Q_1 & P_1 & 0 \\ 0 & P_2 & 0 & Q_2 \end{bmatrix} \right).$$

One can show that $\mathcal{B} = \ker_{\mathcal{L}}[U_4P_1, U_3P_2 + U_4Q_1]$ where $U := \begin{bmatrix} U_1 & U_2 \\ U_3 & U_4 \end{bmatrix} \in Gl_{m+p}(\mathbb{R}[\frac{d}{dt}])$ is such that $U\begin{bmatrix} Q_2 \\ P_1 \end{bmatrix} = \begin{bmatrix} D \\ 0 \end{bmatrix}$ for some full row rank matrix $D \in \mathbb{R}[\frac{d}{dt}]^{m\times m}$, which certainly exists since $\mathbb{R}[\frac{d}{dt}]$ is a principal ideal ring. Moreover, it can easily be seen that the behavior $\mathcal{B}$ is an input/output-system with output y if and only if $\det(I + Q_1^{-1}P_1Q_2^{-1}P_2) \neq 0$, see again [5, Def. 3.3.1, Cor. 3.3.14]. This is the familiar well-posedness condition for this type of feedback-configuration in the classical transfer function approach.

The discussion above raises the following questions:

(Q1) Which subsystems of a given system can be achieved via regular interconnection?

(Q2) Which systems can be achieved from a given system via regular interconnection followed by a projection onto the desired variables?

In the next section we will attack these questions for systems described by delay-differential equations. The characterization for the systems in (Q2) will be new.

3 Interconnection and Elimination for Delay-Differential Systems

In this section we introduce a certain class of delay-differential systems with commensurate delays. This class will also comprise linear systems described by ODEs. Finally, we will discuss the questions (Q1) and (Q2) raised above. We will give algebraic characterizations of the systems under investigation.

For the rest of this paper we will fix the function space for the external variables as $\mathcal{L} := \mathcal{C}^\infty(\mathbb{R}, \mathbb{C})$. Let $D := \frac{d}{dt}$ be the differentiation on $\mathcal{L}$ and let σ denote the forward shift of unit length, i. e. $\big(\sigma f\big)(t) = f(t-1)$ for $f \in \mathcal{L}$ and $t \in \mathbb{R}$. Then the ring

$$\mathbb{R}[D, \sigma] = \Big\{ \sum_{j=0}^{L} \sum_{i=0}^{N} p_{ij} D^i \sigma^j \,\Big|\, L,\, N \in \mathbb{N}_0,\, p_{ij} \in \mathbb{R} \Big\} \subseteq \mathrm{End}_{\mathbb{C}}(\mathcal{L})$$

consists of all linear delay-differential operators with constant coefficients and delays of integral length. If $R \in \mathbb{R}[D, \sigma]^{p\times q}$ is an operator matrix, then $\mathcal{B} := \ker_{\mathcal{L}} R = \{w \in \mathcal{L}^q \mid Rw = 0\}$ is a so-called delay-differential system. In [3] (see also [2]) it has been shown that the action of $\mathbb{R}[D, \sigma]$ on $\mathcal{L}$ is not sufficient in order to allow for a behavioral theory of delay-differential systems. Instead one has to involve more general operators. As to this goal, recall the abstract field of fractions

$$\mathbb{R}(D, \sigma) = \Big\{ \frac{p}{q} \,\Big|\, p,\, q \in \mathbb{R}[D, \sigma],\, q \neq 0 \Big\}$$

of the ring $\mathbb{R}[D,\sigma]$. One should have in mind that its elements have no meaning as operators on $\mathcal{L}$. However, as we will see in Theorem 3.2, a suitably defined subring $\mathcal{H}$ of $\mathbb{R}(D,\sigma)$ will have an interpretation as operator algebra on $\mathcal{L}$. For the rest of this paper the notation $\mathcal{H}$ will be fixed as follows.

Definition 3.1

(a) For $p=\sum_{j,i} p_{ij}D^i\sigma^j \in \mathbb{R}[D,\sigma]$ define the entire function $p^*:\mathbb{C}\to\mathbb{C}$ given by $p^*(s):=p(s,e^{-s})=\sum_{j,i} p_{ij}s^ie^{-js}$ for all $s\in\mathbb{C}$. We call p^* an exponential polynomial.

(b) Define the ring $\mathcal{H}:=\left\{\frac{p}{q}\in\mathbb{R}(D,\sigma)\,\middle|\, p,\,q\in\mathbb{R}[D,\sigma],\, q\neq 0,\,\frac{p^*}{q^*} \text{ is an entire function}\right\}$.

The following results have been shown in [3, Ch. 3].

Theorem 3.2

(1) Each element $\frac{p}{q}\in\mathcal{H}$ defines an operator

$$\frac{p}{q}:\mathcal{L}\longrightarrow\mathcal{L},\quad w\longmapsto p(D,\sigma)v,\ \text{where } q(D,\sigma)v=w$$

and this interpretation yields an embedding of rings $\mathcal{H}\hookrightarrow \operatorname{End}_{\mathbb{C}}(\mathcal{L})$.

(2) $\mathcal{H}$ is an elementary divisor domain, hence every matrix $R\in\mathcal{H}^{p\times q}$ can be brought via left-multiplication (resp. left-right-multiplication) by unimodular matrices into row echelon form (resp. diagonal form).

A few comments are in order. The function p^* associated with the operator $p(D,\sigma)$ is just the characteristic function, hence each zero $\lambda\in\mathbb{C}$ of p^* with multiplicity $k\in\mathbb{N}$ corresponds to the exponential monomials $t^je^{\lambda t}$, $j=0,\dots,k-1$, in the solution space $\ker_{\mathcal{L}}p(D,\sigma)$. Using some facts about the zeros of polynomials in two variables and of exponential polynomials, this yields $\ker_{\mathcal{L}}q(D,\sigma)\subseteq\ker_{\mathcal{L}}p(D,\sigma)$ whenever $\frac{p^*}{q^*}$ is an entire function. Together with the surjectivity of the delay-differential operator $q(D,\sigma)$ [1, p. 697] this leads to the well-definedness of the operator in (1). Part (2) is the result of a thorough algebraic investigation of the ring $\mathcal{H}$. The diagonal form can even be managed as a Smith-form, that is, each diagonal element divides the next one. Therefore, matrices over $\mathcal{H}$ behave basically like matrices with entries in a Euclidean domain like for instance $\mathbb{R}[D]$. In particular, matrices with the same number of columns admit a greatest common right divisor and a least common left multiple, both unique up to left unimodular factors. This strong algebraic structure of $\mathcal{H}$ has the consequence that a behavioral theory for delay-differential systems of the form

$$\mathcal{B}:=\ker_{\mathcal{L}}R \text{ where } R\in\mathcal{H}^{p\times q} \tag{3.1}$$

develops almost parallel to that of linear time-invariant systems described by ODEs as it can be found in the book [5]. For instance, the elimination of latent variables in Example 2.3 work exactly the same way as for systems of ODEs.

It should be noted that an analogous operator algebra can be introduced for systems with noncommensurate delays. However, in that case Theorem 3.2(2) is not true anymore; as a consequence the behavioral theory is much harder than for commensurate delays, see for instance [6].

From now on a *delay-differential system or behavior* will be understood as a system (3.1). As a consequence of the theorem above one has [3, Thm. 4.1.5]

$$\ker_{\mathcal{L}} R_1 \subseteq \ker_{\mathcal{L}} R_2 \Longleftrightarrow \exists\, M \in \mathcal{H}^{p_2 \times p_1} :\ MR_1 = R_2 \tag{3.2}$$

for operator matrices $R_i \in \mathcal{H}^{p_i \times q}$, $i = 1, 2$, and in particular, if $\operatorname{rk} R_i = p_i$, $i = 1, 2$, then

$$\ker_{\mathcal{L}} R_1 = \ker_{\mathcal{L}} R_2 \Longleftrightarrow p_1 = p_2 \text{ and there exists } U \in Gl_{p_1}(\mathcal{H}) :\ UR_1 = R_2. \tag{3.3}$$

Let us present an example of a typical operator in $\mathcal{H}$.

Example 3.3
Let $q = (\sigma - e^{-\lambda})(D - \lambda)^{-1} \in \mathbb{R}(D, \sigma)$. Then $q \in \mathcal{H}$ and we wish to calculate the image $qw \in \mathcal{L}$ for $w \in \mathcal{L}$. In order to do so take $v(t) = \int_0^t e^{\lambda(t-x)} w(x)dx$ as a solution of $(D-\lambda)v = \dot{v} - \lambda v = w$. Then by Theorem 3.2(1) the image $qw \in \mathcal{L}$ is given by

$$qw(t) = (\sigma v - e^{-\lambda} v)(t) = -\int_0^1 e^{\lambda(x-1)} w(t-x)dx.$$

Thus q is a distributed delay operator. The example is typical in the sense that all operators in $\mathcal{H}$ can be written as a linear combination of point delay-differential operators in $\mathbb{R}[D, \sigma]$ and distributed delays similar to the one given above.

Now we are in a position to discuss the questions (Q1) and (Q2) of Section 2. First of all notice that by Theorem 3.2(2) and Equations (3.2) and (3.3) we can always assume that a kernel representation has full row rank and, furthermore, is unique up to left-multiplication by unimodular matrices.

Let us begin with (Q1). The following characterization can be found in [3, Thm. 4.4.4].

Theorem 3.4
Let $R_i \in \mathcal{H}^{p_i \times q}$, $i = 0, 1$ be full row rank matrices and such that $\ker_{\mathcal{L}} R_0 \subseteq \ker_{\mathcal{L}} R_1$. Then the following conditions are equivalent:

(1) $\ker_{\mathcal{L}} R_0$ can be achieved via regular interconnection from $\ker_{\mathcal{L}} R_1$,

(2) $R_0(\ker_{\mathcal{L}} R_1) \subseteq \mathcal{L}^{p_0}$ is a controllable system in the behavioral sense of [5, Def. 5.2.2].

Notice that (2) gives an intrinsic characterization of achievability via regular interconnection; it is purely in terms of the trajectories and does not resort to any kind of systems representation. One should also observe that the set arising in (2) above is always a behavior in the sense of (3.1). More generally, it has been established in [3, Thm. 4.4.1] that for all full row rank operators $R \in \mathcal{H}^{p\times q}$, $\hat{R} \in \mathcal{H}^{\hat{p}\times q}$ one has

$$R(\ker_{\mathcal{L}}\hat{R}) = \ker_{\mathcal{L}}X \Longleftrightarrow XR = \operatorname{lclm}(R, \hat{R}), \tag{3.4}$$

hence the existence of least common left multiples guarantees the existence of some kernel representation $X \in \mathcal{H}^{t\times p}$ for the space $R(\ker_{\mathcal{L}}\hat{R})$.

Using this result we can formulate an answer to question (Q2). In the following theorem the system $\ker_{\mathcal{L}}R_1$ plays the role of the given system, the operator Y that of the projection and the system $\ker_{\mathcal{L}}Z$ is the desired system. Hence part (a) describes $\ker_{\mathcal{L}}Z$ as the projection of a subsystem of $\ker_{\mathcal{L}}R_1$, which in turn can be achieved via regular interconnection. Note also the special case $s = q$ and $Y = I_q$ which leads to Theorem 3.4 again.

Theorem 3.5
Let $R_1 \in \mathcal{H}^{r_1\times q}$ with $\operatorname{rk} R_1 = r_1$, $Y \in \mathcal{H}^{s\times q}$ a unimodular row (that is, $s \leq q$ and Y can be row extended to a matrix in $Gl_q(\mathcal{H})$), and $Z \in \mathcal{H}^{p\times s}$ with $\operatorname{rk} Z = p$. Furthermore, let $X \in \mathcal{H}^{m\times p}$ such that $ZY(\ker_{\mathcal{L}}R_1) = \ker_{\mathcal{L}}X$ (see (3.4)). Then the following are equivalent:

(a) there exists a matrix $R_0 \in \mathcal{H}^{t\times q}$ satisfying

(i) $\operatorname{rk} R_0 = t$,

(ii) $\ker_{\mathcal{L}}Z = Y(\ker_{\mathcal{L}}R_0)$,

(iii) the system $\ker_{\mathcal{L}}R_0$ can be achieved via regular interconnection from $\ker_{\mathcal{L}}R_1$.

(b) the system $\ker_{\mathcal{L}}Z$ can be achieved via regular interconnection from $Y(\ker_{\mathcal{L}}R_1)$.

(c) $\ker_{\mathcal{L}}Z \subseteq Y(\ker_{\mathcal{L}}R_1)$ and $ZY(\ker_{\mathcal{L}}R_1) \subseteq \mathcal{L}^p$ is a controllable behavior.

(d) $\ker_{\mathcal{L}}Z \subseteq Y(\ker_{\mathcal{L}}R_1)$ and X is a unimodular row.

We wish to mention that the matrix X as well as a kernel representation for the behavior $Y(\ker_{\mathcal{L}}R_1)$ can be computed constructively from the data Z, Y, and R_1. Hence the inclusion in part (d) can easily be checked via (3.2), showing that (d) yields a practical criterion. Observe also that (c) provides an intrinsic characterization purely in terms of the trajectories of the systems involved.

Sketch of the Proof: The equivalence (b) $\Leftrightarrow$ (c) follows directly from Theorem 3.4, while (c) $\Leftrightarrow$ (d) is a well-known criterion for controllability [3, Thm. 4.3.8]. As for the equivalence (a) $\Leftrightarrow$ (b) we first notice that there exist full row rank matrices $A \in \mathcal{H}^{a\times s}$ and $M \in \mathcal{H}^{a\times r_1}$ such that

$$AY = MR_1 = \operatorname{lclm}(Y, R_1).$$

This yields $\ker_{\mathcal{L}}A = Y(\ker_{\mathcal{L}}R_1)$, see (3.4). Furthermore the matrix $[A, M]$ is a unimodular row, see [3, Thm. 3.2.8].
(a) $\Rightarrow$ (b): The assumptions together with (3.4) imply the existence of a matrix $\hat{R}_1 \in \mathcal{H}^{l\times q}$

such that $\mathrm{rk}\,[R_1^\mathsf{T}, \hat{R}_1^\mathsf{T}]^\mathsf{T} = r_1 + l$ and

$$ZY = C\begin{bmatrix} R_1 \\ \hat{R}_1 \end{bmatrix} = \mathrm{lclm}\left(Y, \begin{bmatrix} R_1 \\ \hat{R}_1 \end{bmatrix}\right)$$

for some $C \in \mathcal{H}^{p\times(r_1+l)}$. Since $AY = [M, 0]\begin{bmatrix} R_1 \\ \hat{R}_1 \end{bmatrix}$ is also a common left multiple, we get $A = VZ$, $[M, 0] = VC$ for some matrix $V \in \mathcal{H}^{a\times p}$, which has to be a unimodular row, because $[A, M]$ has this property. Hence V can be completed to a unimodular matrix $\begin{bmatrix} V \\ \hat{V} \end{bmatrix} \in Gl_p(\mathcal{H})$. Now $\hat{A} := \hat{V}Z \in \mathcal{H}^{(p-a)\times s}$ satisfies

$$\ker_{\mathcal{L}} Z = \ker_{\mathcal{L}} \begin{bmatrix} A \\ \hat{A} \end{bmatrix}, \tag{3.5}$$

and this establishes (b).

(b) $\Rightarrow$ (a): Let $\hat{A} \in \mathcal{H}^{(p-a)\times s}$ be such that (3.5) is satisfied. Then one can show that $R_0 := \begin{bmatrix} R_1 \\ \hat{A}Y \end{bmatrix}$ establishes (a). □

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