Reduction of Affine Systems on Polytopes

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Abstract

Consider an affine system with a polytope as state set. State trajectories are terminated when they reach a facet of the polytope and attempt to exit. The realization problem is considered based on the behavior of the system, i.e. the set of input-output trajectories on time-intervals of either finite or infinite length. The state set can be affinely reduced due to non-observability if and only if a subspace of the classical unobservable subspace, characterized using the normal vectors of the exit facets, is nontrivial.

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1 Introduction

The purpose of this paper is to present a necessary and sufficient condition for reducibility of an affine system on a polytope. This reduction problem is motivated by control of piecewise-linear hybrid systems. A piecewise-linear hybrid system consists of a finite automaton with at every discrete state of the automaton a continuous-time affine system with a polytope as state set. Control synthesis for this class of hybrid systems and for affine systems on polytopes has been treated elsewhere, see [1, 2]. In this context problems of realization arise, in particular of reachability, observability, and minimality of realizations.

In this paper, the problem is considered whether the state set of an affine system on a polytope can be reduced while the reduced system still is a realization, meaning that it represents the same set of input-output trajectories. An affine system on a polytope has trajectories of infinite length and of finite length; the trajectory terminates when it hits a facet of the polytope and attempts to exit. Realization based on a set of trajectories has been studied before in automata theory, in linear systems, see [5, 6], and in the context of behaviors, see [3]. Novel to realization theory is the simultaneous presence of finite and of infinite length trajectories.

The approach to the problem is to use the concept of unobservable subspaces as used in realization of finite-dimensional linear systems, see [7, 4]. The necessary and sufficient condition for reducibility is then the existence of a nontrivial unobservable subspace of the affine system in the classical sense but restricted by the kernel of the normals of the exit facets. The paper does not discuss the concept of observability of an affine system on a polytope in full generality; this remains to be done.

2 Affine systems on polytopes

Let $N \in \mathbb{N}$, and consider a full-dimensional polytope P_N in $\mathbb{R}^N$, with vertices $v_1, \ldots, v_M$, $(M > N)$. This means that P_N is the convex hull of $\{v_1, \ldots, v_M\}$, and, since P_N is full-dimensional, that there does not exist a hyperplane of $\mathbb{R}^N$ containing all vertices $v_1, \ldots, v_M$. A full-dimensional polytope with exactly $N+1$ vertices is called a full-dimensional *simplex*.

Alternatively, a polytope may be described as the intersection of a finite number of closed half spaces. I.e. there exist an integer $K \geq N+1$, non-zero vectors $n_1, \ldots, n_K \in \mathbb{R}^N$, and scalars $\alpha_1, \ldots, \alpha_K \in \mathbb{R}$, such that

$$P_N = \{x \in \mathbb{R}^N \mid \ \forall i = 1, \ldots, K: \ n_i^T x \leq \alpha_i\}. \tag{2.1}$$

Characterization (2.1) is called the *implicit description* of a polytope. The intersection of a full-dimensional polytope P_N with one of its supporting hyperplanes,

$$F_i := \{x \in \mathbb{R}^N \mid n_i^T x = \alpha_i\} \cap P_N,$$

is called a *facet* of P_N, if the dimension of the intersection is equal to $N-1$. The vector n_i is the normal vector of the facet F_i, $(i = 1, \ldots, K)$, and, by convention, n_i is of unit length and always points out of the polytope P_N.

An *affine map* is a function $f : \mathbb{R}^{N_1} \to \mathbb{R}^{N_2}$ for which there exist $S \in \mathbb{R}^{N_2 \times N_1}$ and $q \in \mathbb{R}^{N_2}$ such that $f(x) = Sx + q$ for all $x \in \mathbb{R}^{N_1}$. Two polytopes $P_1 \subset \mathbb{R}^{N_1}$ and $P_2 \subset \mathbb{R}^{N_2}$ are said to be *affinely isomorphic* if there exists an affine map $f : \mathbb{R}^{N_1} \to \mathbb{R}^{N_2}$ such that P_1 and P_2 are bijectively related by f. It is not required that f is a bijection on $\mathbb{R}^{N_1} \backslash P_1$.

Definition 2.1. A *time-invariant finite-dimensional affine systems on a polytope* (FDAP) is a mathematical structure consisting of a dynamic system defined by the equations,

$$\left\{ \begin{array}{rcl} \dot{x}(t) & = & Ax(t) + Bu(t) + a, \ x(t_0) = x_0, \\ y(t) & = & Cx(t) + Du(t) + c, \end{array} \right. \tag{2.2}$$

with $A \in \mathbb{R}^{N \times N}$, $B \in \mathbb{R}^{N \times m}$, $a \in \mathbb{R}^N$, $C \in \mathbb{R}^{p \times N}$, $D \in \mathbb{R}^{p \times m}$, and $c \in \mathbb{R}^p$. Furthermore, the state x is assumed to be an element of a *full-dimensional polytope* $X \subset \mathbb{R}^N$. Inputs u and outputs y belong to (polyhedral) sets $U \subset \mathbb{R}^m$ and $Y \subset \mathbb{R}^p$, respectively.

For any $x_0 \in X$ and any input trajectory $u : [t_0, \infty) \to U$, the differential equation in (2.2) has a unique solution $x : T_1 \to X$. In order to characterize the property $x \in X$, we adopt the convention that $T_1 = [t_0, \infty)$ if $x(t) \in X$ for all $t \in [t_0, \infty)$, and $T_1 = [t_0, t_1]$ if there exists $t_1 \in [t_0, \infty)$ such that $x(t) \in X$ for all $t \in [t_0, t_1]$ and there exists an $\varepsilon > 0$ such that $x(s) \notin X$ for all $s \in (t_1, t_1 + \varepsilon)$. In the latter case, it is assumed that also input and output trajectories are restricted to the time interval T_1.

An *exit facet* of an affine system is an $(N-1)$-dimensional facet F_i through which the state may leave the state polytope X. More specifically, if F_i is the intersection of X with its supporting hyperplane $\{x \in \mathbb{R}^N \mid n_i^T x = \alpha_i\}$, then it is an exit facet if there exists an input trajectory u and a time instant $t_1 \in \mathbb{R}$, such that the corresponding state trajectory satisfies $x(t) \in X$ for $t \in [t_0, t_1]$ and there exists an $\varepsilon > 0$ such that $n_i^T x(t) > \alpha_i$ for $t \in (t_1, t_1 + \varepsilon)$ (where for a moment it is assumed that system description (2.2) is also valid outside the polytope X). To check whether facet F_i is an exit facet, it suffices to check the velocity vector field $\dot{x}$ at the vertices of the facet: if at one vertex v of F_i there exists an input vector $u \in U$ such that $n_i^T \dot{x} \mid_v = n_i^T (Av + Bu + a) > 0$, then F_i is an exit facet.

Definition 2.2. For a finite-dimensional affine system on a polytope $(FDAP_1)$, the *set of input-output trajectories* for a given set of initial conditions $X_0 \subseteq X$ is defined as

$$IO(FDAP_1, X_0) := \left\{ \begin{array}{l} (u, y) \in U^{T_1} \times Y^{T_1} | \text{either } T_1 = [t_0, \infty) \text{ or } T_1 = [t_0, t_1], \\ \text{and } \exists\ x_0 \in X_0,\ \text{such that } x, y \text{ are solutions of (2.2)} \\ \text{on } T_1 \text{ corresponding to } x_0, u \end{array} \right\}. \tag{2.3}$$

In particular, $IO(FDAP_1, X_0)$ will contain trajectories of both finite and infinite length. If T_1 is infinite, the input-output pair (u, y) admits a state trajectory x that remains in the polytope X forever. If T_1 is finite, the corresponding state trajectory will leave the polytope X at time $t_1 = \max(T_1)$ for the first time.

3 Problem formulation

The realization problem for finite-dimensional affine systems on polytopes will now be formulated in terms of input-output trajectories. This is different from the realization problem for finite-dimensional linear systems, that is formulated for the impulse response function. Special to this case is that the state trajectory may leave the polytope in finite time. Therefore the geometry of the state set and the duration of the input-output trajectories have to be taken into account.

Definition 3.1. Let $U \subseteq \mathbb{R}^m$ and $Y \subseteq \mathbb{R}^p$ be polyhedral sets.

(a) A set of input-output trajectories on these sets is defined as the set,

$$IO \subset \left\{ \begin{array}{l} (u, y) \in U^{T_1} \times Y^{T_1} | \text{either } T_1 = [t_0, \infty) \text{ or } T_1 = [t_0, t_1], \\ \text{such that } u : T_1 \to U,\ y : T_1 \to Y \end{array} \right\}. \tag{3.1}$$

(b) A *realization* of a set IO of input-output trajectories is a finite-dimensional affine system on a polytope $FDAP_1$ and a set of initial states $X_0 \subseteq X$ such that

$$IO = IO(FDAP_1, X_0).$$

Definition 3.2. Consider the subset of finite-dimensional affine systems on a polytope that are realizations of a set of input-output trajectories IO.

(a) Define on this subset the relation $FDAP_2 \leq FDAP_1$, if $\dim(X_2) =: N_2 \leq N_1 := \dim(X_1)$, and if there exists a surjective affine map $f : X_1 \to X_2$ such that for any input-output pair (u, y) all corresponding state trajectories $x_1(t)$ of system $FDAP_1$ have the property that the trajectory defined by $x_2(t) = f(x_1(t))$, $(t \in T_1)$, is a state trajectory of system $FDAP_2$ corresponding to the same input-output pair (u, y).

(b) Define on this subset the relation $FDAP_1 \equiv FDAP_2$ if $N_1 = N_2$ and if there exists a bijective affine transformation $f : X_1 \to X_2$ such that for any input-output pair (u, y) the corresponding state trajectories of these systems are affinely related according to $x_2(t) = f(x_1(t))$ for all $t \in T_1$.

(c) A realization $FDAP_1$ in this subset is said to be *minimal* if there does not exist another realization $FDAP_2$ such that $FDAP_2 \leq FDAP_1$ and $FDAP_1 \not\equiv FDAP_2$.

Problem 3.1. Consider the class of finite-dimensional affine systems on a polytope. Characterize those FDAP-systems that are *minimal* realizations of their associated set of input-output trajectories in the sense described above.

In this paper, minimality of realizations is studied as it was specified in Definition 3.2. In particular, we only consider reduction of the state dimension based on *affine* transformations between the state polytopes. Reductions using more general (i.e. non-affine) transformations may exist, but are not taken into account.

4 Reduction of a realization due to non-observability

In the realization problem for ordinary linear systems, reduction of the dimension of the state space is possible if the system is either not controllable or not observable. In this paper we will extend these ideas to affine systems on polytopes, but limit attention to reduction due to non-observability, i.e. the case $X_0 = X$. In this problem already some new phenomena occur.

Example 4.1. Let $N = 2$ and consider the system $\dot{x}_1(t) = ax_1(t) + u(t)$, $\dot{x}_2(t) = 0$, with output $y(t) = x_1(t)$, where the state $x = (x_1, x_2)^T$ is restricted to the triangle Δ with vertices $v_1 = (-1, 0)^T$, $v_2 = (1, 0)^T$, and $v_3 = (0, 1)^T$. Without the restriction to this simplex, it is clear that the state variable x_2 is neither controllable nor observable, and reduction to a one-dimensional realization is possible. However, by considering the system on the triangle, information on the state variable x_2 becomes available as soon as the evolution of the system stops because the state has left the state polytope. Indeed, since the state x will move only horizontally, it can only leave the triangle through the facet between v_2 and v_3 or the facet between v_3 and v_1. Using the value $y(t_1) = x_1(t_1)$ at the exit time t_1, full information on the value of state variable x_2 is obtained: if $x_1(t_1) \geq 0$, then $x_2(t) = 1 - x_1(t_1)$, and if $x_1(t_1) < 0$, then $x_2(t) = 1 + x_1(t_1)$. As a consequence, the dimension of the state set of this affine system on a triangle cannot be reduced because state variable x_2 is actively involved in the stopping criterion of reaching an exit facet.

The previous example shows that for the observability and reduction of affine systems on polytopes, the geometric structure of the state polytope has to be taken into account. In this section we will present an explicit condition, when reduction of affine systems due to non-observability is possible. In the proof of this result it is necessary to assume that *all* points in the state polytope X may occur as initial state x_0, i.e. $X_0 = X$.

Let $\{F_i \mid i = 1, \ldots, k\}$ denote the set of all exit facets of system (2.2), and assume that $F_i = X \cap \{x \in \mathbb{R}^N \mid n_i^T x = \alpha_i\}$. For $i = 1, \ldots, k$ define $W_i := \ker(n_i^T)$ and correspondingly

$$W := \bigcap_{i=1}^{k} W_i = \ker(N_k),$$

with $N_k = (n_1, n_2, \ldots, n_k)^T \in \mathbb{R}^{k \times n}$. Finally, let V denote the largest A-invariant subspace, contained in $W \cap \ker(C)$, i.e. for $k > 0$:

$$V := \ker\left((C^T \mid N_k^T), A^T(C^T \mid N_k^T), \ldots, (A^T)^{N-1}(C^T \mid N_k^T)\right)^T. \tag{4.1}$$

It will be shown that reduction of the dimension of the state polytope X is possible, using an affine transformation, provided that $V \neq \{0\}$.

Let Π denote the canonical projection on $\mathbb{R}^N/V$. In that situation, there is a commuting diagram of operators, as depicted in Figure 1. Since V is A-invariant, the mapping $\bar{A} : \mathbb{R}^N/V \longrightarrow \mathbb{R}^N/V$, given by $\bar{A}(x + V) = Ax + V$ is well-defined, and satisfies $\bar{A}\Pi = \Pi A$. Similarly, since $V \subset \ker(C)$, the mapping $\bar{C} : \mathbb{R}^N/V \longrightarrow \mathbb{R}^p$, given by $\bar{C}(x + V) = Cx$, is well-defined and $\bar{C}\Pi = C$.

Instead of the original system equations, we now consider the projected system, that is obtained by projecting the state $x \in X$ to $\bar{x} = \Pi x \in \Pi(X)$. The dynamic equations for this projected system are given by

$$\begin{cases} \dot{\bar{x}}(t) &= \bar{A}\bar{x}(t) + \Pi B u(t) + \Pi a, \qquad \bar{x}(t_0) = \Pi x_0, \\ y(t) &= \bar{C}\bar{x}(t) + Du(t) + c, \end{cases} \tag{4.2}$$

with state set $\Pi(X) \subset \mathbb{R}^N/V$.

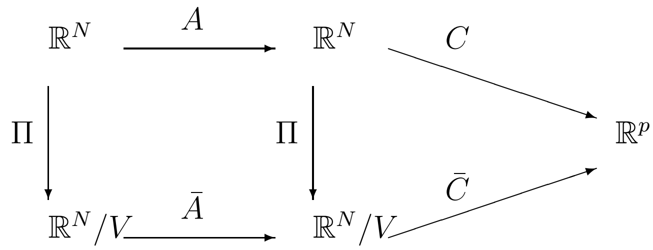

Figure 1: Diagram of commuting operators.

Lemma 4.1. *If X is a full-dimensional polytope in $\mathbb{R}^N$, then $\Pi(X)$ is a full-dimensional polytope in $\mathbb{R}^N/V$. Hence, if $V \neq \{0\}$, then (4.2) characterizes an affine system on a polytope of lower dimension than X.*

The proof of Lemma 4.1 is based on the observation that if the full-dimensional polytope X is described as the convex hull of finitely many points $\{v_1, \dots, v_M\}$, then $\Pi(X)$ is the convex hull of $\{\Pi(v_1), \dots, \Pi(v_M)\}$.

Provided that $V \neq \{0\}$, the projected system (4.2) is our candidate reduced system. If the restrictions that x and $\bar{x}$ have to be elements of the state polytopes X and $\Pi(X)$, respectively, are not taken into account, then $x(t)$ is a state trajectory of system (2.2) corresponding to the input-output pair $(u(t), y(t))$ if and only if $\bar{x}(t) = \Pi x(t)$ is a state trajectory of system (4.2) corresponding to the same input-output pair, because V is a subspace of the unobservable subspace $\langle \ker(C) \mid A \rangle$. In the generalization of this result to systems on polytopes, the normal vectors of the exit facets of X play an important role.

Lemma 4.2. *Let x be a state trajectory of system (2.2) with input trajectory u, and starting in initial state x_0 at time t_0.*

(i) If $x(t) \in X$ for all $t \geq t_0$ (i.e. $T_1 = [t_0, \infty)$), then $\bar{x}(t) = \Pi x(t) \in \Pi(X)$ for all $t \geq t_0$.

(ii) If $T_1 = [t_0, t_1]$, i.e. x leaves the polytope X for the first time at time t_1, then $\bar{x} = \Pi x$ also leaves the polytope $\Pi(X)$ for the first time at time t_1.

Idea of the proof: If for some $t \in \mathbb{R}$, $x(t) \in X$, then it is obvious that $\bar{x}(t) = \Pi x(t) \in \Pi(X)$. The proof that the trajectory $\bar{x}$ leaves $\Pi(X)$ at the same time as trajectory x leaves X is more involved. Let $F_i = \{x \in \mathbb{R}^N \mid n_i^T x = \alpha_i\} \cap X$ be an exit facet of X. Then $V \subset \ker(n_i^T)$, and the mapping $\overline{n_i^T} : \mathbb{R}^N/V \longrightarrow \mathbb{R}$, given by $\overline{n_i^T}(x + V) = n_i^T x$, is well-defined and satisfies $\overline{n_i^T}\Pi = n_i^T$. Since $n_i^T x \leq \alpha_i$ for all $x \in X$, also $\overline{n_i^T}\bar{x} \leq \alpha_i$ for all $\bar{x} \in \Pi(X)$. Next, assume that $x(t)$ attempts to leave X through exit facet F_i at time t_1. Then there exists $\varepsilon > 0$ such that $n_i^T x(t) > \alpha_i$ for all $t \in (t_1, t_1 + \varepsilon)$. Hence also $\overline{n_i^T}\bar{x}(t) = \overline{n_i^T}\Pi x(t) = n_i^T x(t) > \alpha_i$ for $t \in (t_1, t_1 + \varepsilon)$, which implies that also $\bar{x}(t) \notin \Pi(X)$ for $t \in (t_1, t_1 + \varepsilon)$. ∎

Corollary 4.1. *The original system (2.2) and the projected system (4.2) realize the same input-output trajectories, both of finite and infinite length, provided that in both systems the initial state may be chosen arbitrarily in X and $\Pi(X)$ respectively. In particular, if $V \neq \{0\}$ then system (2.2) may be reduced to system (4.2) using the canonical projection Π as the affine reduction map.*

Finally it is shown that $V \neq \{0\}$ is not only a sufficient but also a necessary condition for reduction.

Proposition 4.1. *Consider system (2.2) and assume that there exists an affine map f, reducing system (2.2) (in the sense of Definition 3.2) to a realization*

$$\left\{ \begin{array}{rcl} \dot{x}_1(t) & = & A_1 x_1(t) + B_1 u(t) + a_1, \quad x_1(t_0) = x_{1,0}, \\ y(t) & = & C_1 x_1(t) + D_1 u(t) + c_1. \end{array} \right. \tag{4.3}$$

I.e. system (4.3) is an affine system on a full-dimensional polytope X_1 of dimension $N_1 < N$, and admits the same set of input-output trajectories as system (2.2). Then $V \neq \{0\}$.

Proof: Let the affine map f reducing (2.2) to (4.3) be given by $f: X \longrightarrow X_1: \ f(x) = Sx + q$, with $S \in \mathbb{R}^{N_1 \times N}$ of full row rank and $q \in \mathbb{R}^{N_1}$. We show that $\{0\} \neq \ker(S) \subset V$.

Let $(u(t), y(t))$ be an arbitrary pair of input-output trajectories, and assume that $x(t)$ is a corresponding state trajectory of system (2.2). Then $x_1(t) = Sx(t) + q$ is a corresponding state trajectory of system (4.3) and we have

$$\dot{x}_1(t) = S\dot{x}(t) = S(Ax(t) + Bu(t) + a) = SAx(t) + SBu(t) + Sa$$

on the one hand, and

$$\dot{x}_1(t) = A_1 x_1(t) + B_1 u(t) + a_1 = A_1 S x(t) + B_1 u(t) + (A_1 q + a_1)$$

on the other. For $u(t_0) = 0$, and since $x(t_0) = x_0$ may be chosen arbitrarily in the full-dimensional polytope X, we conclude that $SA = A_1 S$ (*).

Similarly, since

$$\begin{array}{rcl} y(t) & = & Cx(t) + Du(t) + c = \\ & = & C_1 x_1(t) + D_1 u(t) + c_1 = C_1 S x(t) + D_1 u(t) + (C_1 q + c_1), \end{array}$$

and using the same arguments as above, we find $C_1 S = C$ (**).

Next, let $n_i^T x = \alpha_i$ ($i = 1, \ldots, k$) denote the hyperplanes containing the exit facets of system (2.2), and let $m_j^T x_1 = \beta_j$ ($j = 1, \ldots, \ell$) denote the hyperplanes containing the exit facets of system (4.3). If a trajectory $x(t)$ of (2.2) leaves X at time t_1, then the corresponding trajectory $x_1(t) = Sx(t) + q$ leaves X_1 also at time t_1. In combination with a continuity argument, this implies that the affine map f maps the exit facets of (2.2) to exit facets of (4.3), i.e. for all $i \in \{1, \ldots, k\}$ there exists a $j \in \{1, \ldots, \ell\}$ such that all $x \in \mathbb{R}^N$ satisfying

$n_i^T x = \alpha_i$ also satisfy $m_j^T(Sx+q) = \beta_j$. Since $n_i \neq 0$ and $m_j^T S \neq 0$, the equations $n_i^T x = \alpha_i$ and $m_j^T Sx = \beta_j - m_j^T q$ describe the same hyperplane, so for all $i \in \{1, \dots, k\}$ there exists a $c_i \neq 0$ and a $j \in \{1, \dots, \ell\}$ such that $n_i^T = c_i m_j^T S$ (***).

(*) implies that $\ker(S)$ is an A-invariant subspace, (**) indicates that $\ker(S) \subset \ker(C)$, and (***) shows that $\ker(S) \subset \cap_{i=1}^k \ker(n_i^T) = W$. So $\ker(S)$ is an A-invariant subspace contained in $\ker(C) \cap W$, hence $\ker(S) \subset V$. Since $\ker(S) \neq \{0\}$, also $V \neq \{0\}$. ∎

Theorem 4.1. *FDAP-system (2.2) with the full state polytope X as set of initial conditions is a minimal realization in the sense of Definition 3.2 if and only if $V = \{0\}$, where V denotes the subspace as defined in (4.1).*

Example 4.2. The system in Example 4.1 is a minimal realization of its input-output behavior because $V = \{0\}$. Indeed, the normal vectors to the exit facets of the polytope are $n_1 = (1,1)^T$ and $n_2 = (-1,1)^T$, and thus $V \subset \ker(n_1^T) \cap \ker(n_2^T) = \{0\}$.

Example 4.3. Consider the system $\dot{x}_1(t) = ax_1(t) + u(t)$, $\dot{x}_2(t) = 0$ on the square $-1 \leq x_1 \leq 1$, $-1 \leq x_2 \leq 1$, with output $y(t) = x_1(t)$. Now the exit facets are segments of the lines $x_1 = 1$ and $x_1 = -1$ with normal vectors $n_1 = (1,0)^T$ and $n_2 = (-1,0)^T$, respectively. Since $\ker(C) \cap \ker(n_1^T) \cap \ker(n_2^T) = \langle (0,1)^T \rangle$ is A-invariant, $V = \langle (0,1)^T \rangle$. Therefore this realization is not minimal, and may be reduced in dimension by deleting the state variable x_2.

5 Concluding remarks

A realization of a set of input-output trajectories of an affine system on a polytope can be affinely reduced if and only if a particular subspace generated by the unobservable subspace and the null space of the normals of the exit facets, is nontrivial. This realization problem is of interest to control of piecewise-linear hybrid systems.

Further research on the realization problem of affine systems on polytopes is required. If the set of initial states is not equal to the full state polytope, the problem becomes much harder because the reachable subset of such an affine system is not necessarily a polytope.

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