Some New Results on Linear Quadratic Regulator Design for Lossless Systems

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Abstract

In this paper some new results concerning linear quadratic regulator (LQR) design for lossless systems are proved. The first result gives a necessary and sufficient condition which allows to obtain directly the optimal gains of an LQR problem for a lossless system, without solving the CARE equation, when a particular quadratic index is considered. The given results can also be viewed as an inverse LQR design problem. Moreover, given a system with simple purely imaginary poles, the conditions which allow to compute the gain matrix of an LQR problem without solving the Riccati equation are derived.

1 Introduction

Even if both the LQR topic [2], [3] and the peculiarity of passive systems and in particular, those of lossless systems [1], [4], are widely faced in the literature, the lossless property referred to the LQR control techniques is not deeply investigated [5]. In this paper some interesting results are obtained. In particular in the next section a new theorem is proved which allows to obtain the optimal gains of an LQR problem for a lossless system without solving the CARE equation. The optimal gain are obtained directly from the output matrix of the the lossless system and the weight matrix of the LQR performance index.
Moreover a set of conditions which allow to obtain the optimal gains of a given LQR problem without solving the CARE equation are stated for systems with simple purely imaginary poles.

2 LQR design for lossless systems

Before introducing the main results of the paper some theorems and definitions are needed. Let us consider a *n-th* order square MIMO system in a minimal form *S(A,B,C)*.

Let $G(s)=C(sI-A)^{-1}B$ be its transfer matrix. The necessary and sufficient condition for $S(A,B,C)$ to be a lossless positive system are the following:

- All poles of $G(s)$ are simple and have zero real part and the residue matrix at any pole is a nonnegative definite matrix;
- $G^T(-j\omega)+G(j\omega)=0$ for all ω real such that $j\omega$ is not a pole of any element of $G(s)$.

Moreover for a lossless system the following Positive Real Lemma holds [6]:
PR Lemma
A system $S(A,B,C)$ is lossless iff it exists a symmetric, positive definite matrix P_1 such that:

$$P_1A + A^TP_1 = 0 \tag{2.1}$$

$$P_1B = C^T \tag{2.2}$$

The LQR problem that will be considered in this paper is the minimization of the index J defined as:

$$\begin{aligned} &J = \int_0^\infty \left(y^TQy + u^TQ^{-1}u\right) dt = \int_0^\infty \left(x^TC^TQCx + u^TQ^{-1}u\right)dt \\ &Q > 0 \end{aligned} \tag{2.3}$$

where x are the state variables and y is the output vector.
The control law is given as $u=-Kx$ where K is the optimal gain matrix.
Even if the considered problem, due to the special form of the weighting matrices, is a restriction of the general LQR problem, it includes a large set of LQR problems and in particular the normalised weight $Q = I$, in which the results reported in this paper have an interesting meaning.
The main result of the paper is expressed by the following theorem.

Theorem 2.1 *Let us consider a lossless system $S(A,B,C)$, then the optimal gain matrix K of the LQR problem stated by (2.3) is defined by the following equality:*

$$K = QC \tag{2.4}$$

and:

$$u=-QCx=-Qy \tag{2.5}$$

therefore the gain matrix K is directly derived without solving the Riccati Equation or using Hamiltonian properties. Moreover if relation (2.4) holds, the system is lossless.

Proof:
Let us prove the theorem by using the Positive Real Lemma for lossless systems.
For hypotesis $S(A,B,C)$ is a lossless system, therefore conditions (2.1) and (2.2) holds.
Let us verify that matrix P_1 satisfies the control Riccati equation (CARE):

$$A^T P + PA - PBQB^T P + C^T QC = 0 \qquad (2.6)$$

which minimises index (2.3), where the matrix of the optimal gain is:

$$K = QB^T P$$

Then, substituting $P = P_1$ equation (2.6) becomes, by using (2.1):

$$-P_1 BQB^T P_1 + C^T QC = 0$$

and by using (2.2) we obtain that P_1 is a solution of (2.6) and therefore:

$$K = QB^T P_1 = QC$$

To prove the necessary condition let us suppose that for a given system $S(A, B, C)$ we have $K = QB^T P = QC$ where P is the symmetric, positive definite, solution of (2.6). By substituting this condition in (2.6) we obtain:

$$A^T P + PA = 0$$

conditions (2.1) and (2.2) are therefore satisfied and $S(A, B, C)$ is lossless.

This proves the introduced theorem.

The previous problem can be also reconsidered as an inverse LQ problem as shown in the following statement.

Theorem 2.2 *Given a lossless system S(A, B, C) and the static output feedback law u=-Qy with* $Q > 0$ *then this feedback law is a solution of an LQR problem with performance index (2.3).*

Remarks

It should be observed that if $Q = I$ we have $K = C$ and the control law is therefore $u = -y$. Moreover if there are pertubations on the matrix A that maintain the system passive the closed loop system remains robustly stable with the control law *u=-Qy.*

In the following, the conditions under which the gain matrix of an LQR problem can be determined without solving the CARE equation are given.

Theorem 2.3 *Given a system S(A, B) with n couples of simple purely imaginary eigenvalues, it exists a family of matrices C so that the system S(A, B, C) is lossless.*

Let us prove the theorem.

Given the matrix A it exist a tranformation matrix T so that

$$\tilde{A} = T * A * T^{-1} = \begin{bmatrix} 0 & -\omega_1 & 0 & 0 & 0 & 0 \\ \omega_1 & 0 & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & -\omega_n \\ 0 & 0 & 0 & 0 & \omega_n & 0 \end{bmatrix}$$

For the matrix $\tilde{A}$ it holds:

$$P^* \tilde{A} + \tilde{A}^T P^* = 0$$

with:

$$P^* = \begin{bmatrix} \gamma_1 I & 0 & 0 \\ \cdots & \cdots & \cdots \\ 0 & 0 & \gamma_n I \end{bmatrix}$$

where I is the 2 dimensional identity matrix and γ_i are n parameters, and therefore:

$$P^* T A T^{-1} + (T A T^{-1})^T P^* = 0$$

Multiplying the equation above by T^T on the left and by T on the right and by denoting $P = T^T P^* T$ we obtain:

$$PA + A^T P = 0$$

if we define:

$C = PB$ we obtain a family of lossless systems $S(A, B, C)$ and the theorem is proved. The parameters γ_i are n free parameters that can be selected to design the weight matrix of the LQR index.

Corollary 2.1 *Given a system S(A, B) with n couples of simple purely imaginary eigenvalues, the LQR problem with performance index:*
$J = \int\limits_0^\infty \left(x^T Q x + u^T u \right) dt$
where $Q = C^T C > 0$, and x are the state variables, is solved by the optimal control law u=-Cx.

The proof of the corollary follows directly from Theorem 2.1 and 2.3.

3 Conclusions

In this paper it has been proven that for lossless systems the optimal LQR gain matrix is directly obtained as the product of the weight matrix and the output matrix.

In the special case of identity weight matrix, the optimal control law is therefore $u = -y$. Moreover, given a system with simple purely imaginary eigenvalues a corresponding family of lossless systems for which the LQR design problem can be solved directly can be obtained.

4 References

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