On Minimal Order Decentralized Output Feedback Pole Assignment Problems

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Abstract

We give a new sufficient condition for the arbitrary pole assignability of a system by decentralized dynamic compensators. Using such condition we are able to derive a new bound on the degrees of decentralized dynamic compensators so that the generic systems have the arbitrary pole assignability.

1 Introduction

For many large scale systems like electric power systems, transportation systems and whole economic systems, it is desirable to decentralize the control task. This is in particular preferable if the measurements have been taken on decentralized local channels and the controls can be applied on local channels only. Decentralized stabilization and pole assignment of linear systems have been studied by many authors. Wang and Davison [12] proved that decentralized stabilization using local dynamic compensators is possible if and only if the fixed modes are stable. Corfmat and Morse [2] proved that a strongly connected system can be made controllable and observable through a single channel by local static feedback if and only if the set of fixed modes is empty. Thus the poles of such a system can be assigned freely. The control strategy of [2] is to apply local static feedback to all but one channel, in order to make the resulting single channel system controllable. Then a dynamic feedback controller is applied to the channel to assign the closed-loop poles. Wang [13] proved that if the dimension of the set of all static local compensators is greater than the McMillan degree of the system, then almost all r-channel systems having the same number of inputs or outputs on the local channels are arbitrarily pole assignable by decentralized static output feedback. Ravi, Rosenthal, and Wang [7] introduce a parameterization of the set of all decentralized dynamic compensators, and proved that the decentralized pole assignment map is onto over $\mathbb{C}$ for generic systems as soon as the dimension of the set of decentralized dynamic compensators is greater than or equal to the total number of the closed-loop poles. However such result is not true over $\mathbb{R}$ as demonstrated by Willems and Hesselink for the centralized case [16].

The problem we are interested in is to find minimal order decentralized dynamic compensators to stabilize or to assign the poles of a given system. We give a new sufficient condition which ensures arbitrary pole assignability for a given system. Using such condition we are able to derive a new bound on the degrees of decentralized dynamic compensators so that a generic system has the arbitrary pole assignability.

2 Preliminary Results

We first define some terminologies about the polynomial matrices. Let $M(s)$ be a $p\times(m+p)$ polynomial matrix over $\mathbb{R}$ with $m>0$. The i^{th} *row degree* of $M(s)$ is defined as the highest polynomial degree among all the entries in the i^{th} row. The *high degree coefficient matrix* of $M(s)$, denoted by M_h, is defined to be the matrix consisting of the coefficients of the monomials whose degrees equal the corresponding row degrees. The *McMillan degree* of a full rank, non-square polynomial matrix is defined to be the highest degree of its full size minors. A matrix $M(s)$ is called *row proper* if M_h has full rank, and it is called *irreducible* if the full size minors of $M(s)$ are relatively prime. $M(s)$ is called *minimal* if its rows form a minimal basis of the row space; i.e. they form a basis, and the sum of the row degrees is minimal among all the bases of the row space. It has been proved in [4] that a $p\times(m+p)$ polynomial matrix $M(s)$ is minimal if, and only if, it is row proper and irreducible. The row degrees of a minimal basis of the row space of $M(s)$ are called the *Forney indices* of $M(s)$.

Similar terminologies are also defined for $(m+p)\times p$ matrices if we interchange "row" and "column". Let $M(s)$ be a minimal matrix. A *dual matrix* of $M(s)$, denoted by $M^{\perp}(s)$, is an $(m+p)\times m$ minimal polynomial matrix such that

$$M(s)M^{\perp}(s)=0.$$

The Forney indices of $M^{\perp}(s)$ are called the *dual Forney indices* of $M(s)$. The sum of the Forney indices equals the sum of the dual Forney indices [4].

Proposition 2.1. *[5] Let $\mathcal{P}$ be the set of all $p\times(m+p)$ polynomial matrices of row degrees $(\mu_1,\ldots,\mu_p)$. Set $n=\mu_1+\cdots+\mu_p$, and let $k=\lfloor n/m\rfloor$ be the largest integer $\le n/m$, and $d=n-km$ be the remainder of n divided by m. There exists a nonempty Zariski open set $\mathcal{S}\subset\mathcal{P}$ of minimal matrices such that*

1. *every matrix $M(s)$ in $\mathcal{S}$ has the dual Forney indices*

$$\nu_g:=\{\underbrace{k,\ldots,k}_{m-d},\underbrace{k+1,\ldots,k+1}_{d}\},\tag{2.1}$$

and

2. *for all $M(s)\in\mathcal{S}$, the coefficients of the polynomials in $M^{\perp}(s)$ are rational functions, with nonzero denominators, of coefficients of the polynomials in $M(s)$.*

The set of all unimodular column equivalence classes of $(m+p)\times m$ irreducible polynomial matrices of McMillan degree n is a quasi projective variety [8]. In this quasi projective variety, the equivalence classes with the Forney indices ν_g defined in (2.1) form a nonempty Zariski open set, and the set of all the other equivalence classes has strictly smaller dimension [9]. For this reason we call the Forney indices ν_g defined in (2.1) the *generic dual indices* of $\mathcal{P}$.

We formulate the problem in the behavior framework [15]. Consider an r-channel linear system

$$\dot{x} = Ax + \sum_{i=1}^{r} B_i u_i, \quad y_i = C_i x + \sum_{j=1}^{r} D_{ij} u_j, \quad i = 1, 2, \ldots, r \tag{2.2}$$

where x, u_i, y_i are n, m_i, p_i vectors, respectively, and u_i and y_i are the input and output of the ith channel.

Let

$$m = m_1 + \cdots + m_r, \quad p = p_1 + \cdots + p_r.$$

Write the system as

$$\begin{bmatrix} \frac{d}{dt}I - A & 0 & -B_1 & 0 & -B_2 & \cdots & 0 & -B_r \\ C_1 & -I & D_{11} & 0 & D_{12} & \cdots & 0 & D_{1r} \\ C_2 & 0 & D_{21} & -I & D_{22} & \cdots & 0 & D_{2r} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ C_r & 0 & D_{r1} & 0 & D_{r2} & \cdots & -I & D_{2r} \end{bmatrix} \begin{bmatrix} x \\ y_1 \\ u_1 \\ y_2 \\ u_2 \\ \vdots \\ y_r \\ u_r \end{bmatrix} = 0. \tag{2.3}$$

If the system is observable, then there is a unimodular polynomial matrix $U(s)$ such that

$$U(s) \begin{bmatrix} \frac{d}{dt}I - A \\ C_1 \\ \vdots \\ C_r \end{bmatrix} = \begin{bmatrix} I \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Multiplying (2.4) by $U(\frac{d}{dt})$ from left, we then have an equivalent system

$$\begin{bmatrix} I & * \\ 0 & P(\frac{d}{dt}) \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} = 0, \tag{2.4}$$

where

$$w = \begin{bmatrix} w_1 \\ \vdots \\ w_r \end{bmatrix} \text{ with } w_i = \begin{bmatrix} y_i \\ u_i \end{bmatrix}$$

and the $*$ represents something unimportant.

Under behavior framework,

$$P(\frac{d}{dt})w = 0 \tag{2.5}$$

is called the kernel representation of the system. If in addition, the system is also observable, then the McMillan degree of $P(s)$ equals n, and by applying elementary row operations if necessary, we can always assume that the $p \times m + p$ matrix $P(s)$ is minimal. In this case, there is also an image representation of the system. Since $P(s)$ is minimal, there exists a unimodular matrix $U(s)$ such that $P(s)U(s) = [I, 0]$. Let

$$U(s) = [\hat{Q}(s), Q(s)]$$

where $Q(s)$ is $(m + p) \times m$, and

$$\begin{bmatrix} \hat{v} \\ v \end{bmatrix} = U^{-1}(\frac{d}{dt})w$$

where $v \in \mathbb{R}^m$. Then (2.5) becomes

$$\hat{v} = 0,$$

and

$$w = Q(\frac{d}{dt})v \tag{2.6}$$

is called image representation of the system.

Let the decentralized dynamic compensators in the image representation be given by

$$w_i = Q_i(\frac{d}{dt})v_i, \quad i = 1, \ldots, r, \tag{2.7}$$

where each $Q(s)$ is $(m_i + p_i) \times p_i$ minimal polynomial matrix of McMillan degree q_i (notice that the inputs of the ith compensator are the outputs of the ith local channel, which is in $\mathbb{R}^{p_i}$).

The closed-loop system becomes

$$P(\frac{d}{dt}) \begin{bmatrix} Q_1(\frac{d}{dt}) & & \\ & \ddots & \\ & & Q_r(\frac{d}{dt}) \end{bmatrix} \begin{bmatrix} v_1(t) \\ \vdots \\ v_r(t) \end{bmatrix} = 0, \tag{2.8}$$

and the closed-loop poles are given by the zeros of

$$\det \left(P(s) \begin{bmatrix} Q_1(s) & & \\ & \ddots & \\ & & Q_r(s) \end{bmatrix} \right)$$

provided it is not a zero polynomial.

Definition 2.2. We say an r-tuple decentralized compensators $\{Q_i(s)\}$ dependent if

$$\det \left(P(s) \begin{bmatrix} Q_1(s) & & \\ & \ddots & \\ & & Q_r(s) \end{bmatrix} \right)$$

is a zero polynomial.

Crucial to our study is the decentralized pole assignment map. We only define it for a generic set of decentralized dynamic compensators of degrees $q_1, \ldots, q_r$. Let

$$k_i = \lfloor q_i/p_i \rfloor$$

be the largest integer less than or equal to q_i/p_i,

$$d_i = q_i - k_i p_i$$

be the remainder of q_i divided by p_i, and let $\mathcal{G}$ be the set of all r-tuple polynomial matrices $Q(s) = \text{diag }(Q_1(s), \ldots, Q_r(s))$ such that each $Q_i(s)$ is $(m_i + p_i) \times p_i$ with column degrees at most

$$\underbrace{k_i, \ldots, k_i}_{p_i - d_i}, \underbrace{k_i + 1, \ldots, k_i + 1}_{d_i}.$$

Define the pole assignment map $\chi : \mathcal{G} \to \mathbb{R}^{n+q_1+\cdots+q_r+1}$

$$\chi(Q(s)) = \det\left(P(s)Q(s)\right) \tag{2.9}$$

Proposition 2.3. *The pole assignment map is onto if there is a r-tuple dependent decentralized compensators $Q(s) \in \mathcal{G}$ such that the Jacobin $d\chi_Q$ is onto.*

Proof. χ maps a neighborhood of $Q(s)$ to a neighborhood of origin by the inverse function theorem. Since χ is homogenous, χ is onto the whole $\mathbb{R}^{n+q_1+\cdots+q_r+1}$. □

Proposition 2.4. *[10, Theorem 3.10] The Jacobian $d\chi_Q : \mathcal{G} \to \mathbb{R}^{n+q_1+\cdots+q_r+1}$ is given by*

$$d\chi_Q(X(s)) = \text{tr}(R(s)X(s))$$

where

$$R(s) = \text{adj}(P(s)Q(s))P(s).$$

3 Main Results

In this section we give a new bound for the degrees of pole assigning decentralized compensators for the generic systems.

Theorem 3.1. *A r-tuple decentralized dependent compensators of degrees at most $(q_1, \ldots, q_r)$ exists if either*

$$q_i + (\lfloor q_i/p_i \rfloor + 1)(m_i - 1) \geq \lfloor n/p \rfloor, \quad i = 1, \ldots, r, \tag{3.1}$$

or

$$q_i + (\lfloor q_i/m_i \rfloor + 1)(p_i - 1) \geq \lfloor n/m \rfloor, \quad i = 1, \ldots, r. \tag{3.2}$$

Proof. We only need to prove one of the cases. The second case can be proved similarly if we consider the image representation of the system and kernel representation the compensator.

Assume

$$q_i + (\lfloor q_i/p_i \rfloor + 1)(m_i - 1) \geq \lfloor n/p \rfloor, \quad i = 1, \dots, r.$$

Let

$$P(s) = \begin{bmatrix} \alpha(s) \\ \hat{P}(s) \end{bmatrix}$$

be minimal of dgree n, where $\alpha(s)$ is a row of the smallest degree. Write

$$\alpha(s) = (\alpha_1(s), \dots, \alpha_r(s))$$

where each $\alpha_i(s)$ is a $(m_i + p_i)$-vector. Then row deg $\alpha_i(s) \leq \lfloor n/p \rfloor$.

We construct a decentralized dependent compensator as follows: For each $\alpha_i(s)$, let

$$\alpha_i^{\perp}(s) = [f_1(s), \dots, f_{m_i+p_i-1}(s)]$$

where

$$\deg f_1 \leq \cdots \leq \deg f_{m_i+p_i-1}.$$

Define

$$Q_i(s) = [f_1(s), \dots, f_{p_i}(s)].$$

Then

$$(Q_1(s), \dots, Q_r(s))$$

is a r-tuple decentralized dependent compensators.

The McMillian degree of $Q_i(s)$ is $\deg f_1 + \cdots + \deg f_{p_i}$, and we claim that it is at most q_i. If not

$$\deg f_1 + \cdots + \deg f_{p_i} > q_i$$

then $\deg f_{p_i} \geq \lfloor q_i/p_i \rfloor + 1$ so

$$\begin{aligned} \lfloor n/p \rfloor &\geq \deg \alpha_i(s) \\ &\geq \deg f_1 + \cdots + \deg f_{m_i+p_i-1} \\ &> q_i + (\lfloor q_i/p_i \rfloor + 1)(m_i - 1) \end{aligned}$$

which contradicts the condition. □

If such $Q(s) \in \mathcal{G}$, then the Jacobin $d\chi_Q$ has a very simple form.

Lemma 3.2. *Let*

$$P(s) = \begin{bmatrix} \alpha(s) \\ \hat{P}(s) \end{bmatrix}.$$

and $\alpha(s)Q(s) = 0$. *Then*

$$d\chi_Q(X(s)) = \det \begin{bmatrix} \alpha(s)X(s) \\ \hat{P}(s)Q(s) \end{bmatrix}.$$

Proof. Since $\alpha(s)Q(s) = 0$, it follows that only the first column of $\text{adj}(P(s)Q(s))$ is nonzero. Let it be $\eta(s)$. Then

$$R(s) = \text{adj}(P(s)Q(s))P(s) = \eta(s)\alpha(s)$$

and

$$d\chi_Q(X(s)) = \text{tr}(R(s)X(s)) = \alpha(s)X(s)\eta(s) = \det \begin{bmatrix} \alpha(s)X(s) \\ \hat{P}(s)Q(s) \end{bmatrix}.$$

□

Theorem 3.3. *A system $P(s)$ has the arbitrary pole assignability by decentralized dynamic compensators of degrees at most $(q_1, \ldots, q_r)$ where*

$$q_i + (\lfloor q_i/p_i \rfloor + 1)(m_i - 1) \geq \lfloor n/p \rfloor, \quad i = 1, \ldots, r$$

if

1. *$P(s)$ has the generic Forney indices*

$$(\underbrace{l, \ldots, l}_{p-e}, \underbrace{l+1, \ldots, l+1}_{e}), \quad l = \lfloor n/p \rfloor, \quad e = n - lp;$$

2. *Each $\alpha_i(s)$ has the generic dual Forney indices,*

$$(\underbrace{\epsilon, \ldots, \epsilon}_{m_i+p_i-1-\delta}, \underbrace{\epsilon+1, \ldots, \epsilon+1}_{\delta}), \quad \epsilon = \lfloor l/(m_i + p_i - 1) \rfloor, \ \delta = l - \epsilon(m_i + p_i - 1)$$

where $\alpha(s) = (\alpha_1(s), \ldots, \alpha_r(s))$ is a row of $P(s)$ of the smallest degree with each $\alpha_i(s) \in \mathbb{R}^{m_i+p_i}$;

3. *$\hat{P}(s)Q(s)$ is irreducible with McMillan degree*

$$n + q_1 + \cdots + q_r - \lfloor n/p \rfloor - \min\lfloor q_i/p_i \rfloor,$$

where

$$Q(s) = \text{diag}(Q_1(s), \ldots, Q_r(s))$$

and each $(m_i + p_i) \times p_i$ matrix $Q_i(s)$ consists of p_i columns of $\alpha_i(s)^\perp$ of the smallest column degrees, and $\hat{P}(s)$ is the $(p-1) \times (m+p)$ matrix formed by removing $\alpha(s)$ from $P(s)$.

Proof. By Lemma 3.2, we need to show that the linear map

$$d\chi_Q(X(s)) = \det \begin{bmatrix} \alpha_1(s)X_1(s), \ldots, \alpha_r(s)X_r(s) \\ \hat{P}(s)Q(s) \end{bmatrix}$$

is onto the space of all polynomials of degree at most $n + q_1 + \cdots q_r$.

Let the column degrees of $Q(s) = \mathrm{diag}(Q_1(s), \ldots, Q_r(s))$ be

$$(\nu_1, \ldots, \nu_p),$$

and define

$$\mathcal{Z} = \{z(s) = (z_1(s), \ldots, z_p(s)) \mid \deg z_j(s) \leq l + \nu_j\}.$$

We write the linear map $d\chi_Q$ into a composition of two linear maps:

1. $\phi : \mathcal{G} \to \mathcal{Z}$ defined by

$$\phi(X(s)) = (\alpha_1(s)X_1(s), \ldots, \alpha_r(s)X_r(s)),$$

   and

2. $\psi : \mathcal{Z} \to \mathbb{R}^{n+q_1+\cdots+q_r+1}$ defined by

$$\psi(z(s)) = \det \begin{bmatrix} z(s) \\ \hat{P}(s)Q(s) \end{bmatrix},$$

and show both of them are onto.

Let

$$X_i(s) = [x_{\xi_i+1}, \ldots, x_{\xi_i+p_i}], \quad \xi_1 = 0,\ \xi_i = p_1 + \cdots p_{i-1}.$$

To show that ϕ is onto, it is enough to show that

$$\phi_j(x_j) := \alpha_i x_j, \quad \xi_i + 1 \leq j \leq \xi_i + p_i$$

is onto the space of polynomials of degree at most $l + \nu_j$ for all the x_j of column degrees at most ν_j.

By [1] ϕ_j has a rank

$$(m_i + p_i)(\nu_j + 1) - \sum_{\mu_s \leq \nu_j} (\nu_j + 1 - \mu_s)$$

where $\{\mu_s\}$ are the dual Forney indices of $\alpha_i(s)$. When $\nu_j = \epsilon$

$$\sum_{\mu_s \leq \nu_j} (\nu_j + 1 - \mu_s) = m_i + p_i - 1 - \delta$$

and when $\nu_j = \epsilon + 1$

$$\sum_{\mu_s \leq \nu_j} (\nu_j + 1 - \mu_s) = 2(m_i + p_i - 1) - \delta.$$

In both cases

$$\mathrm{rank}\ \phi_j = l + \nu_j + 1,$$

and therefore ϕ_j is onto.

Since the dimension of $\mathcal{Z}$ is $p(l+1)+q_1+\cdots+q_r$ and the dimension of the range of ψ is $n+q_1+\cdots+q_r+1$, in order to show that ψ is onto, it is sufficient to show that

$$\dim\ker\psi = p(l+1)-n-1 = p-e-1.$$

By re-arrange of rows if necessary, we can assume that $\hat{P}(s)$ has row degrees

$$(\underbrace{l,\ldots,l}_{p-1-e},\underbrace{l+1,\ldots,l+1}_{e}).$$

If $z(s)\in\ker\psi$, then by [4]

$$z(s) = [a_1(s),\ldots,a_{p-1}(s)]\hat{P}(s)Q(s)$$

for some polynomials $\{a_j(s)\}$. We claim that $a_j(s)=a_j$ for $j\le p-1-e$, and $a_j(s)=0$ for $j>p-1-e$. If not, then

$$\deg y(s) := \deg[a_1(s),\ldots,a_{p-1}(s)]\hat{P}(s) > l.$$

Note that the given conditions imply that $\hat{P}_hQ_h$ has full rank $p-1$, because its full size minors are the coefficients of the monomials of $s^{n+q_1+\cdots+q_r-\lfloor n/p\rfloor-\min\lfloor q_i/p_i\rfloor}$ of the corresponding full size minors of $\hat{P}(s)Q(s)$, and therefore not all of them are zero. Since $y_h\in$ row space$\hat{P}_h$, we must have $y_hQ_h\neq 0$. Assume that j-th entry of y_hQ_h is nonzero. Then $\deg z_j(s) > l+\nu_j$ and $z(s)\notin\mathcal{Z}$. Therefore

$$\ker\psi \subset \{[a_1,\ldots,a_{p-1-e},0,\ldots,0]\hat{P}(s)Q(s)\}$$

and $\dim\ker\psi\le p-1-e$. □

Remark 3.4. The Forney indices of $P(s)$ are the observability indices of the system.

Finally since the conditions of Theorem 3.3 are satisfied by generic systems (see [5]), we have

Theorem 3.5. *The closed loop poles of generic m-input, p-output, system of McMillian degree n can be arbitrarily assigned by a decentralized dynamic compensator of McMillian degrees $(q_1,\ldots,q_r)$ if either*

$$q_i + (\lfloor q_i/p_i\rfloor+1)(m_i-1) \ge \lfloor n/p\rfloor,\quad i=1,\ldots,r,$$

or

$$q_i + (\lfloor q_i/m_i\rfloor+1)(p_i-1) \ge \lfloor n/m\rfloor,\quad i=1,\ldots,r.$$

where m_i,p_i are the numbers of inputs and outputs, respectively, for the ith local channel.

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